PREFACE TO PART I.

The work, of which the part now issued is a first instal-
ment, has been compiled from notes made at various periods
of the last fourteen years, and chiefly during the engagements
of teaching. Many of the abbreviated methods and mnemonic
rules are in the form in which I originally wrote them for my
pupils.

The general object of the compilation is, as the title
indicates, to present within a moderate compass the funda-
mental theorems, formulæ, and processes in the chief branches
of pure and applied mathematics.

The work is intended, in the first place, to follow and
supplement the use of the ordinary text-books, and it is
arranged with the view of assisting the student in the task of
revision of book-work. To this end I have, in many cases,
merely indicated the salient points of a demonstration, or
merely referred to the theorems by which the proposition is
proved. I am convinced that it is more beneficial to the
student to recal[illegible] demonstrations with such aids, than to read
[illegible]d re-read the[illegible] [illegible] them be read once, but recalled often.
[illegible] [illegible]on the mind between reading a[illegible]
[illegible] [illegible]inating one wholly or

A SYNOPSIS

OF

ELEMENTARY RESULTS

Part I. of Volume I., now ready, contains the following sections, which may be purchased separately, in stiff paper covers, at the prices annexed:—

		s.	d.
Section I.	—MATHEMATICAL TABLES; including Measures in C. G. S. units. Burckhardt's Table of Least Factors of numbers from 1 to 99000. Legendre's Values of the Gamma-function, &c. .... Price	2	0
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Price of Part I. complete, Cloth, Royal 8vo, 280 pages		12	0

Part II. of Volume I., which is in the Press, will contain—

Section VIII.—DIFFERENTIAL CALCULUS.
„ IX.—INTEGRAL CALCULUS.
„ X.—CALCULUS OF VARIATIONS.
„ XI.—DIFFERENTIAL EQUATIONS.
„ XII.—PLANE COORDINATE GEOMETRY.
„ XIII.—SOLID COORDINATE GEOMETRY.

Volume II. is in preparation, and will be devoted to APPLIED MATHEMATICS and other BRANCHES OF PURE MATHEMATICS.

LONDON:
C. F. HODGSON AND SON, GOUGH SQUARE,
FLEET STREET.

1880.

A SYNOPSIS

OF

ELEMENTARY RESULTS

IN

PURE AND APPLIED MATHEMATICS:

CONTAINING

PROPOSITIONS, FORMULÆ, AND METHODS OF ANALYSIS,

WITH

ABRIDGED DEMONSTRATIONS.

BY

G. S. CARR, B.A.,

LATE PRIZEMAN AND SCHOLAR, OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

VOL. I.

LONDON:

C. F. HODGSON AND SON, GOUGH SQUARE,

FLEET STREET.

1880.

1814–17, which give the least divisors of all numbers from 1 to 3,036,000. In a certain sense, it may be said that this is the only sort of purely mathematical table which is absolutely indispensable, because the information which it gives cannot be supplied by any process of direct calculation. The logarithm of a number, for instance, may be computed by a formula. Not so its prime factors. These can only be arrived at through the tentative process of successive divisions by the prime numbers, an operation of a most deterrent kind when the subject of it is a high integer.

A table similar to and in continuation of Burckhardt's has recently been constructed for the fourth million by J. W. L. Glaisher, F.R.S., who I believe is also now engaged in completing the fifth and sixth millions. The factors for the seventh, eighth, and ninth millions were calculated previously by Dase and Rosenberg, and published in 1862–65, and the tenth million is said to exist in manuscript. The history of the formation of these tables is both instructive and interesting.*

As, however, such tables are necessarily expensive to purchase, and not very accessible in any other way to the majority of persons, it seemed to me that a small portion of them would form a useful accompaniment to the present volume. I have, accordingly, introduced the first eleven pages of Burckhardt's tables, which give the least factors of the first 100,000 integers nearly. Each double page of the table here printed is

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* See "*Factor Table for the Fourth Million.*" By James Glaisher, F.R.S. London: Taylor and Francis. 1880. Also *Camb. Phil. Soc. Proc.*, Vol. III., Pt. IV., and *Nature*, No. 542, p. 462.

an exact reproduction, in all but the type, of a single quarto page of Burckhardt's great work.

It may be noticed here that Prof. Lebesque constructed a table to about this extent, on the plan of omitting the multiples of seven, and thus reducing the size of the table by about one-sixth.* But a small calculation is required in using the table which counterbalances the advantage so gained.

The values of the Gamma-Function, pages 30 and 31, have been taken from Legendre's table in his "*Exercices de Calcul Intégral*," Tome I. The table belongs to Part II. of this Volume, but it is placed here for the convenience of having all the numerical tables of Volume I. in the same section.

In addition to the authors already named, the following treatises have been consulted—Algebras, by Wood, Bourdon, and Lefebure de Fourcy; Snowball's Trigonometry; Salmon's Higher Algebra; the Geometrical Exercises in Potts's Euclid; and Geometrical Conics by Taylor, Jackson, and Renshaw.

Articles 260, 431, 569, and very nearly all the examples, are original. The latter have been framed with great care, in order that they might illustrate the propositions as completely as possible.

G. S. C.

HADLEY, MIDDLESEX;
May 23, 1880.

* "Tables diverses pour la décomposition des nombres en leurs facteurs premiers." Par V. A. Lebesque. Paris. 1864.

ERRATA.

Art. 13,		*for*	$-a^2b^2$	*read*	$+a^2b^2$.
,, 56,	Line 1,	,,	3	,,	$\frac{3}{2}$.
,, 66,	,, 5,	,,	x	,,	x^2.
,, 90,	,, 4,	,,	numerators 1, 1, 1	,,	$1, a, a^2$.
,, 99,	,, 1,	,,	denominator $r-1$	,,	$n-1$.
,, 107,	,, 1,	,,	taken	,,	taken m at a time.
,, 108,	,, 2,	,,	(196)	,,	(360).
,, 131,	,, 1, 2,	,,	5	,,	6.
,, ,,	,, 5,	,,	$(-1)^5$	,,	$(-1)^{23}$.
,, 133,	,, 3, 6, 7,	,,	$6x$	,,	$3x$.
,, ,,	,, 8,	,,	4	,,	34.
,, ,,	,, 9,	,,	204, 459	,,	102, 306.
,, ,,	,, 10,	,,	459	,,	$9n$.
,, 138,	,, 4,	,,	$\frac{10.9.8}{1.2.3}$	,,	7.8.9.10.
,, 140,		,,	$(q+1)^k$	,,	$(q+1)^{(k)}$ Notation of (96).
,, 182,	,, 5,	,,	u_{n-1} in numerator	,,	u^2_{n-1}.
,, 191,	,, 4,	,,	(163)	,,	(164).
,, 220,	,, 6,	,,	$(x+y+z)^2$	,,	$2(x+y+z)^2$.
,, 221,	,, 4,	,,	(1)	,,	square of (1).
,, 237,	,, 11,	,,	$x^2=1$	,,	$x^2=-1$.
,, 238,	,, 5,	,,	(x^2-4x+8) on left side	,,	$(x^2-4x+8)^3$.
,, 239,	,, 11,		(234)		*Dele.*
,, 248,	,, 4,	,,	(29)	,,	(28).
,, 267,	,, 4,	,,	(267)	,,	(266).
,, 274,	,, 8,	,,	$\lfloor 11$	,,	$2\lfloor 11$.
,, 276,	,, 13,	,,	$p+2$	,,	$p+1$.
,, ,,	,, 14,	,,	$(p-1)$	,,	$\lfloor p-1$.
,, 283,	,, 3,	,,	$x=1$	,,	$a=b$.
,, 288,	,, 7,	,,	$n-1$	,,	$n+1$.
,, 289,	,, 4,	,,	$H(r, n-1)$	,,	$H(n, r-1)$.
,, 290,	,, 2,	,,	$H(r+1, n-1)$	,,	$H(n, r)$.
,, 325,	,, 17,	,,	P_2	,,	P.
,, ,,		,,	$P_1P_2P_3$, last line but one	,,	$Q_1Q_2Q_3$.
,, 333,	,, 3,	,,	$\left(\frac{a+b}{2}\right)$	,,	$\left(\frac{a+b}{2}\right)^m$.
,, 361,	,, 7,	,,	3528	,,	10284.
,, 481,	,, 6,	,,	$n-3$	,,	$n-1$.
,, 514,	,, 4,		applying Descartes' rule		*Dele.*
,, 517,	,, 3,	,,	a^3	,,	x^3.
,, 544,	,, 1,				Transpose F and f.
,, 551,	,, 1,	,,	B_1	,,	B.
,, ,,	,, 9,	,,	$a-n$	,,	$a-\kappa$.
,, 704,		,,	(11, 12)	,,	(9, 10, 1).
,, 729,		,,	(940)	,,	(960).

Article 112 should be as follows:—

$$\frac{1}{1+2\sqrt{3}+\sqrt{2}}=\frac{1+2\sqrt{3}+\sqrt{2}}{(1+2\sqrt{3})^2-2}=\frac{1+2\sqrt{3}+\sqrt{2}}{11+4\sqrt{3}}=\frac{(1+2\sqrt{3}+\sqrt{2})(11-4\sqrt{3})}{73}.$$

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SECTION I.—MATHEMATICAL TABLES.

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INDEX TO PROPOSITIONS OF EUCLID

REFERRED TO IN THIS WORK.

The references to Euclid are made in Roman and Arabic numerals; *e.g.* (VI. 19).

BOOK I.

I. 4.—Triangles are equal and similar if two sides and the included angle of each are equal each to each.

I. 5.—The angles at the base of an isosceles triangle are equal.

I. 6.—The converse of 5.

I. 8.—Triangles are equal and similar if the three sides of each are equal each to each.

I. 16.—The exterior angle of a triangle is greater than the interior and opposite.

I. 20.—Two sides of a triangle are greater than the third.

I. 26.—Triangles are equal and similar if two angles and one corresponding side of each are equal each to each.

I. 27.—Two straight lines are parallel if they make equal alternate angles with a third line.

I. 29.—The converse of 27.

I. 32.—The exterior angle of a triangle is equal to the two interior and opposite; and the three angles of a triangle are equal to two right angles.

Cor. 1.—The interior angles of a polygon of n sides $= (n-2)\,\pi$.

Cor. 2.—The exterior angles $= 2\pi$.

I. 35 to 38.—Parallelograms or triangles upon the same or equal bases and between the same parallels are equal.

I. 43.—The complements of the parallelograms about the diameter of a parallelogram are equal.

I. 47.—The square on the hypotenuse of a right-angled triangle is equal to the squares on the other sides.

I. 48.—The converse of 47.

BOOK II.

II. 4.—If a, b are the two parts of a right line, $(a+b)^2 = a^2+2ab+b^2$.

If a right line be bisected, and also divided, internally or externally, into two unequal segments, then—

II. 5 and 6.—The rectangle of the unequal segments is equal to the difference of the squares on half the line, and on the line between the points of section; or $(a+b)(a-b) = a^2-b^2$.

II. 9 and 10.—The squares on the same unequal segments are together double the squares on the other parts; or

$$(a+b)^2+(a-b)^2 = 2a^2+2b^2.$$

II. 11.—To divide a right line into two parts so that the rectangle of the whole line and one part may be equal to the square on the other part.

II. 12 and 13.—The square on the base of a triangle is equal to the sum of the squares on the two sides *plus* or *minus* (as the vertical angle is *obtuse* or *acute*), twice the rectangle under either of those sides, and the projection of the other upon it; or $a^2 = b^2+c^2-2bc\cos A$ (702).

BOOK III.

III. 3.—If a diameter of a circle bisects a chord, it is perpendicular to it: and conversely.

III. 20.—The angle at the centre of a circle is twice the angle at the circumference on the same arc.

III. 21.—Angles in the same segment of a circle are equal.

III. 22.—The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

III. 31.—The angle in a semicircle is a right angle.

III. 32.—The angle between a tangent and a chord from the point of contact is equal to the angle in the alternate segment.

III. 33 and 34.—To *describe* or to *cut off* a segment of a circle which shall contain a given angle.

III. 35 and 36.—The rectangle of the segments of any chord of a circle drawn through an *internal* or *external* point is equal to the square of the semi-chord perpendicular to the diameter through the internal point, or to the square of the tangent from the external point.

III. 37.—The converse of 36. If the rectangle be equal to the square, the line which meets the circle touches it.

BOOK IV.

IV. 2.—To inscribe a triangle of given form in a circle.
IV. 3.—To describe the same about a circle.
IV. 4.—To inscribe a circle in a triangle.
IV. 5.—To describe a circle about a triangle.
IV. 10.—To construct two-fifths of a right angle.
IV. 11.—To construct a regular pentagon.

BOOK VI.

VI. 1.—Triangles and parallelograms of the same altitude are proportional to their bases.

VI. 2.—A right line parallel to the side of a triangle cuts the other sides proportionally; and conversely.

VI. 3 and A.—The bisector of the *interior* or *exterior* vertical angle of a triangle divides the base into segments proportional to the sides.

VI. 4.—Equiangular triangles have their sides proportional homologously.

VI. 5.—The converse of 4.

VI. 6.—Two triangles are equiangular if they have two angles equal, and the sides about them proportional.

VI. 7.—Two triangles are equiangular if they have two angles equal and the sides about two *other* angles proportional, provided that the third angles are both greater than, both less than, or both equal to a right angle.

VI. 8.—A right-angled triangle is divided by the perpendicular from the right angle upon the hypotenuse into triangles similar to itself.

VI. 14 and 15.—Equal *parallelograms*, or *triangles* which have two angles equal, have the sides about those angles reciprocally proportional; and conversely, if the sides are in this proportion, the figures are equal.

VI. 19.—Similar triangles are in the duplicate ratio of their homologous sides.

VI. 20.—Likewise similar polygons.

VI. 23.—Equiangular parallelograms are in the ratio compounded of the ratios of their sides.

VI. B.—The rectangle of the sides of a triangle is equal to the square of the bisector of the vertical angle *plus* the rectangle of the segments of the base.

BOOK XI.

MATHEMATICAL TABLES.

INTRODUCTION.

The Centimetre-Gramme-Second system of units.

NOTATION.—The decimal measures of length are the *kilometre, hectometre, decametre, metre, decimetre, centimetre, millimetre*. The same prefixes are used with the *litre* and *gramme* for measures of capacity and volume.

Also, 10^7 metres is denominated a *metre-seven;* 10^{-7} metres, a *seventh-metre;* 10^{15} grammes, a *gramme-fifteen;* and so on.

A gramme-million is also called a *megagramme;* and a millionth-gramme, a *microgramme;* and similarly with other measures.

DEFINITIONS.—The C. G. S. system of units refers all physical measurements to the *Centimetre* (cm.), the *Gramme* (gm.), and the *Second* (sec.) as the units of length, mass, and time.

The quadrant of a meridian is approximately a *metre-seven*. More exactly, one metre = 3·2808694 feet = 39·370432 inches.

The *Gramme* is the *Unit of mass*, and the *weight of a gramme* is the *Unit of weight*, being approximately the weight of a cubic centimetre of water; more exactly, 1 gm. = 15·432349 grs.

The *Litre* is a cubic decimetre: but *one cubic centimetre* is the C. G. S. *Unit of volume.*

1 litre = ·035317 cubic feet = ·2200967 gallons.

The *Dyne* (dn.) is the *Unit of force*, and is the force which, in one second generates in a gramme of matter a velocity of one centimetre per second.

The *Erg* is the *Unit of work and energy*, and is the work done by a dyne in the distance of one centimetre.

The absolute *Unit of atmospheric pressure* is one megadyne per square centimetre = 74·964 cm., or 29·514 in. of mercurial column at 0° at London, where g = 981·17 dynes.

Elasticity of Volume = k, is the pressure per unit area upon a body divided by the cubic dilatation.

Rigidity $= n$, is the shearing stress divided by the angle of the shear.

Young's Modulus $= M$, is the longitudinal stress divided by the elongation produced, $= 9nk \div (3k+n)$.

Tenacity is the tensile strength of the substance in dynes per square centimetre.

The *Gramme-degree* is the *Unit of heat*, and is the amount of heat required to raise by 1° C. the temperature of 1 gramme of water at or near 0°.

Thermal capacity of a body is the increment of heat divided by the increment of temperature. When the increments are small, this is the thermal capacity *at* the given temperature.

Specific heat is the thermal capacity of unit mass of the body at the given temperature.

The *Electrostatic unit* is the quantity of electricity which repels an equal quantity at the distance of 1 centimetre with the force of 1 dyne.

The *Electromagnetic unit* of quantity $= 3 \times 10^{10}$ *electrostatic units* approximately.

The *Unit of potential* is the potential of unit quantity at unit distance.

The *Ohm* is the common *electromagnetic unit* of resistance, and is approximately $= 10^9$ *C. G. S. units*.

The *Volt* is the *unit of electromotive force*, and is $= 10^8$ *C. G. S. units of potential.*

The *Weber* is the *unit of current*, being the current due to an electromotive force of 1 Volt, with a resistance of 1 Ohm. It is $= \frac{1}{10}$ *C. G. S. unit.*

Resistance of a Wire = Specific resistance × Length ÷ Section.

Physical constants and Formulæ.

In the latitude of London, $g = 32\cdot19084$ feet per second.
$= 981\cdot17$ centimetres per second.

In latitude λ, at a height h above the sea level,

$$g = (980\cdot6056 - 2\cdot5028 \cos 2\lambda - \cdot000003\,h) \text{ centimetres per second.}$$

Seconds pendulum $= (99\cdot3562 - \cdot2536 \cos 2\lambda - \cdot0000003\,h)$ centimetres.

THE EARTH.—Semi-polar axis, 20,854895 feet* $= 6\cdot35411 \times 10^8$ centims.
Mean semi-equatorial diameter, 20,926202 „ * $= 6\cdot37824 \times 10^8$ „
Quadrant of meridian, 39·377786 × 10^7 inches* $= 1\cdot000196 \times 10^7$ metres.
Volume, 1·08279 cubic centimetre-nines.
Mass (with a density $5\frac{1}{2}$) = Six gramme-twenty-sevens nearly.

* These dimensions are taken from Clarke's "Geodesy," 1880.

Velocity in orbit = 2933000 centims. per sec. Obliquity, 23° 27′ 16″.*
Angular velocity of rotation = 1 ÷ 13713.
Precession, 50″·26.* Progression of Apse, 11″·25. Eccentricity, e = ·01679.
Centrifugal force of rotation at the equator, 3·3912 dynes per gramme.
Force of attraction upon moon, ·2701. Force of sun's attraction, ·5839.
Ratio of g to centrifugal force of rotation, $g : r\omega^2 = 289$.
Sun's horizontal parallax, 8″·7 to 9′.* Aberration, 20″·11 to 20″·79.*
Semi-diameter at earth's mean distance, 16′ 1″·82.*
Approximate mean distance, 92,000000 miles, or 1·48 centimetre-thirteens.†
Tropical year, 365·242216 days, or 31,556927 seconds.
Sidereal year, 365·256374 ,, 31,558150 ,,
Anomalistic year, 365·259544 days. Sidereal day, 86164 seconds.

THE MOON.—Mass = Earth's mass × ·011364 = 6·98 10^{25} grammes.
Horizontal parallax. From 53′ 56″ to 61′ 24″.*
Sidereal revolution, 27d. 7h. 43m. 11·46s. Lunar month, 29d. 12h. 44m. 2·87s.
Greatest distance from the earth, 251700 miles, or 4·05 centimetre-tens,
Least ,, ,, 225600 ,, 3·63 ,, ,,
Inclination of Orbit, 5° 9′. Annual regression of Nodes, 19° 20′.

RULE.—(*The Year*+1) ÷ 19. *The remainder is the Golden Number.*
(*The Golden Number*−1) × 11 ÷ 30. *The remainder is the Epact.*

GRAVITATION.— Attraction between masses m, m' at a distance l $= \dfrac{mm'}{l^2 \times 1{\cdot}543 \times 10^7}$ dynes.

The mass which at unit distance (1 cm.) attracts an equal mass with unit force (1 dn.) is $= \sqrt{(1{\cdot}543 \times 10^7)}$ gm. = 3928 gm.

WATER.—Density at 0° C., unity; at 4°, 1·000013 (Kupffer).
Volume elasticity at 15°, $2{\cdot}22 \times 10^{10}$.

Compression for 1 megadyne per sq. cm., $4{\cdot}51 \times 10^{-5}$ (Amaury and Descamps).

The heat required to raise the temperature of a mass of water from 0° to t° is proportional to $t + {\cdot}00002t^2 + {\cdot}0000003t^3$ (Regnault).

GASES.—Expansion for 1° C., ·003665 = 1 ÷ 273.

$$\frac{\text{Specific heat at constant pressure}}{\text{Specific heat at constant volume}} = 1{\cdot}408.$$

Density of dry air at 0° with Bar. at 76 cm. = ·0012932 gm. per cb. cm. (Regnault).
At unit pres. (a megadyne) Density = ·0012759.
Density at press. $p = p \times 1{\cdot}2759 \times 10^{-9}$.

Density of saturated steam at t°, with p taken from Table II., is approximately $= \dfrac{{\cdot}7936098p}{(1 + {\cdot}00365t)\, 10^9}$.

SOUND.—Velocity = √(*elasticity of medium* ÷ *density*).
Velocity in dry air at t° = 33240 $\sqrt{(1 + {\cdot}00366t)}$ centimetres per second.
Velocity in water at 0° = 143000 ,, ,,

LIGHT.—Velocity in a medium of absolute refrangibility μ
$= 3{\cdot}004 \times 10^{10} \div \mu$ (Cornu).

If P be the pressure in dynes per sq. cm., and t the temperature,
$\mu - 1 = 2903 \times 10^{-13} P \div (1 + {\cdot}00366t)$ (Biot & Arago).

* These data are from the "Nautical Almanack" for 1883.
† Transit of Venus, 1874, "Astrom. Soc. Notices," Vols. 37, 38.

Table I.

Various Measures and their Equivalents in C. G. S. units.

Dimensions.

1 inch	= 2·5400 cm.
1 foot	= 30·4797 „
1 mile	= 160933 „
1 nautical do.	= 185230 „
1 sq. inch	= 6·4516 sq. cm.
1 sq. foot	= 929·01 „
1 sq. yard	= 8361·13 „
1 sq. mile	= $2{\cdot}59\times10^{10}$ „
1 cb. inch	= 16·387 cb. cm.
1 cb. foot	= 28316 „
1 cb. yard	= 764535 „
1 gallon	= 4541 „
	= 277·274 cb. in. or the volume of 10 lbs. of water at 62° Fah., Bar. 30 in.

Mass.

1 grain	= ·06479895 gm.
1 ounce	= 28·3495 „
1 pound	= 453·5926 „
1 ton	= 1,016047 „
1 kilogramme	= 2·20462125 lbs.
1 pound Avoir.	= 7000 grains
1 pound Troy	= 5760 „

Velocity.

1 mile per hour	= 44·704 cm. per sec.
1 kilometre „	= 27·77̇ „

Pressure.

1 gm. per sq. cm.	= 981 dynes per sq. cm.
1 lb. per sq. foot	= 479 „
1 lb. per sq. in.	= 68971 „
76 centimetres of mercury at 0° C.	= 1,014,000 „

$$\frac{\text{lbs. per sq. in.}}{\text{gms. per sq. cm.}} = 70{\cdot}307 = \frac{1}{{\cdot}014223}$$

Force of Gravity.

upon 1 gramme	= 981	dynes
„ 1 grain	= 63·56777	„
„ 1 oz.	= $2{\cdot}7811\times10^{4}$	„
„ 1 lb.	= $4{\cdot}4497\times10^{5}$	„
„ 1 cwt.	= $4{\cdot}9837\times10^{7}$	„
„ 1 ton	= $9{\cdot}9674\times10^{8}$	„

Work ($g=981$).

1 gramme-centimetre	= 981	ergs.
1 kilogram-metre	= 981×10^{5}	„
1 foot-grain	= $1{\cdot}937\times10^{8}$	„
1 foot-pound	= $1{\cdot}356\times10^{7}$	„
1 foot-ton	= $3{\cdot}04\times10^{10}$	„
1 'horse power' p. sec.	= $7{\cdot}46\times10^{9}$	„

Heat.

1 gramme-degree C.	= 42×10^{6}	ergs.
1 pound-degree	= 191×10^{8}	„
1 pound-degree Fah.	= 106×10^{8}	„

Table II.

Pressure of Aqueous Vapour in dynes per square centim.

Temp.	Pressure.	Temp.	Pressure.
−20°	1236	40°	73200
−15°	1866	50°	122600
−10°	2790	60°	198500
− 5°	4150	80°	472900
0°	6133	100°	1014000
5°	8710	120°	1988000
10°	12220	140°	3620000
15°	16930	160°	6210000
20°	23190	180°	10060000
25°	31400	200°	15600000
30°	42050		

Table III.

Values for the principal Lines of the Spectrum in air at 160° C. with Bar. 76 cm.

	Wave-length in centims.	No. of vibrations per second.
A	$7{\cdot}604\times10^{-5}$	$3{\cdot}950\times10^{14}$
B	6·867 „	4·373 „
C	6·56201 „	4·577 „
D (mean)	5·89212 „	5·097 „
E	5·26913 „	5·700 „
F	4·86072 „	6·179 „
G	4·30725 „	6·973 „
H_1	3·96801 „	7·569 „
H_2	3·93300 „	7·636 „

TABLE IV. *in C. G. S. units.*

	Density Water = 1.	Young's Modulus *M*.	Rigidity *n*.	Elasticity of volume *k*.	Tenacity.	Expansion of Volume per degree C.	Linear Expansion between 0 & 100 C.	Specific Heat between 0 & 100 C.	Relative Conductivity.	Rate of Conduction of Sound in cm. per sec.	Elect. Magn. Specific Resistance at 0° C.
Platinum	21	—	—	—	—	·000027	·000875	·0335	381	$2·69 \times 10^5$	9158
Gold	19·26	—	—	—	—	·000045	·001483	—	1000	1·74 ,,	2081
Mercury	13·596	—	—	$0·542 \times 10^{12}$	—	·000180	—	·0330	—	—	96190
Lead	11·35	$·059 \times 10^{12}$	—	—	$2·28 \times 10^8$	·000086	·002861	—	180	1·23 ,,	19850
Silver	10·47	—	—	—	—	·000061	·00196	·0557	973	2·61 ,,	1521
Copper	8·843	1·234 ,,	$4·47 \times 10^{11}$	1·684 ,,	41·4 ,,	·000054	·00175	·0949	898	3·74 ,,	1615
Brass, drawn	8·471	1·075 ,,	3·66 ,,	—	33·8 ,,	·000054	·00193	—	357	3·56 ,,	—
Iron, cast	7·235	1·349 ,,	5·32 ,,	0·964 ,,	—	·000033	·001111	—	—	4·32 ,,	—
Iron, wrought	7·677	1·963 ,,	7·69 ,,	1·456 ,,	58·6 ,,	·000040	·001258	·1098	374	5·06 ,,	9827
Steel	7·849	2·139 ,,	8·19 ,,	1·841 ,,	79·3 ,,	·000037	·001260	—	—	5·22 ,,	—
Tin, cast	7·29	—	—	—	3·17 ,,	·000063	·00227	—	304	—	13360
Zinc, cast	7·19	—	—	—	5·17 ,,	·000088	·00294	·0927	363	—	5690
Glass, flint	2·942	0·603 ,,	2·40 ,,	0·415 ,,	—	·000015	·00081	·1770	—	4·53 ,,	—

TABLE V.

	Greatest distance from Sun. Earth's mean distance = 1.	Least distance from Sun.	Sidereal Revolution in Days.	Inclination of Orbit to Ecliptic.			Time of Rotation.			Diameter in Miles.	Mass.	Density.
				°	′	″	h.	m.	s.			
Sun	—	—	—		—		600	0	0	888000	354936·	0·25
Mercury	0·46669	0·30750	87·969	7	0	8	24	5	28	3000	0·118	2·01
Venus	0·72826	0·71840	224·701	3	23	31	23	21	21	7700	0·883	0·97
Earth	1·01678	0·98322	365·256		—		23	56	4	7926	1·000	1·00
Mars	1·66578	1·38160	686·980	1	51	5	24	37	22	4500	0·132	0·72
Jupiter	5·45378	4·95182	4332·585	1	18	40	9	55	26	92000	338·034	0·24
Saturn	10·07328	9·00442	10759·220	2	29	28	10	29	17	75000	101·064	0·13
Uranus	20·07612	18·28916	30686·821		46	30		—		36000	14·789	0·15
Neptune	30·29888	29·77506	60126·722	1	46	59		—		35000	24·648	0·27

Table VI.—*Functions of π and e.*

π = 3·1415926	π^{-1} = ·3183099	e = 2·71828183
π^2 = 9·8696044	π^{-2} = ·1013212	e^2 = 7 38905611
π^3 = 31·0062761	π^{-3} = ·0322515	e^{-1} = 0·3678794
$\sqrt{\pi}$ = 1·7724539	$200^g \div \pi$ = 63^g·6619772	e^{-2} = 0·1353353
$\log_{10}\pi$ = 1·4971499	$180^\circ \div \pi$ = 57°·2957795	$\log_{10} e$ = 0·43429448
$\log_e \pi$ = 0·6679358	= 206264″·8	$\log_e 10$ = 2·30258509

Table VII.

No.	Square root.	Cube root.
2	1·4142136	1·2599210
3	1·7320508	1·4422496
4	2·0000000	1·5874011
5	2·2360680	1·7099759
6	2·4494897	1·8171206
7	2·6457513	1·9129312
8	2·8284271	2·0000000
9	3·0000000	2·0800837
10	3·1622777	2·1544347
11	3·3166248	2·2239801
12	3·4641016	2·2894286
13	3·6055513	2·3513347
14	3·7416574	2·4101422
15	3·8729833	2·4662121
16	4·0000000	2·5198421
17	4·1231056	2·5712816
18	4·2426407	2·6207414
19	4·3588989	2·6684016
20	4·4721360	2·7144177
21	4·5825757	2·7589243
22	4·6904158	2·8020393
23	4·7958315	2·8438670
24	4·8989795	2·8844991
25	5·0000000	2·9240177
26	5·0990195	2·9624960
27	5·1961524	3·0000000
28	5·2915026	3·0365889
29	5·3851648	3·0723168
30	5·4772256	3·1072325

Table VIII.

N.	$\log_{10} N$.	$\log_e N$.
2	·3010300	·69314718
3	·4771213	1·09861229
5	·6989700	1·60943791
7	·8450980	1·94591015
11	1·0413927	2·39789527
13	1·1139434	2·56494936
17	1·2304489	2·83321334
19	1·2787536	2·94443898
23	1·3617278	3·13549422
29	1·4623980	3·36729583
31	1·4913617	3·43398720
37	1·5682017	3·61091791
41	1·6127839	3·71357207
43	1·6334685	3·76120012
47	1·6720979	3·85014760
53	1·7242759	3·97029191
59	1·7708520	4·07753744
61	1·7853298	4·11087386
67	1·8260748	4·20469262
71	1·8512583	4·26267988
73	1·8633229	4·29045944
79	1·8976271	4·36944785
83	1·9190781	4·41884061
89	1·9493900	4·48863637
97	1·9867717	4·57471098
101	2·0043214	4·61512052
103	2·0128372	4·63472899
107	2·0293838	4·67282883
109	2·0374265	4·69134788

Note.—The authorities for Table IV. are as follows:—Columns 2, 3, and 4 (Mercury excepted), Everett's experiments (Phil. Trans., 1867); g is here taken = 981·4. The densities in these cases are those of the specimens employed. Cols. 5 and 7, Rankine. Col. 6, Watt's Dict. of Chemistry. Col. 8, Dulong and Petit. Col. 10, Wertheim. Col. 11, Matthiessen.

Table V. is abridged from Loomis's Astronomy.

The values in Table III. are Angström's.

BURCKHARDT'S FACTOR TABLES.

For all numbers from 1 to 99000.

Explanation.—The tables give the least divisor of every number from 1 up to 99000: but numbers divisible by 2, 3, or 5 are not printed. All the digits of the number whose divisor is sought, excepting the units and tens, will be found in one of the three rows of larger figures. The two remaining digits will be found in the left-hand column. The least divisor will then be found in the column of the first named digits, and in the row of the units and tens.

If the number be prime, a cipher is printed in the place of its least divisor.

The numbers in the first left-hand column are not consecutive. Those are omitted which have 2, 3, or 5 for a divisor. Since $2^2.3.5^2 = 300$, it follows that this column of numbers will re-appear in the same order after each multiple of 300 is reached.

Mode of using the Tables.—If the number whose prime factors are required is divisible by 2 or 5, the fact is evident upon inspection, and the division must be effected. The quotient then becomes the number whose factors are required. If this number, being within the range of the tables, is yet not given, *it is divisible by* 3. Dividing by 3, we refer to the tables again for the new quotient and its least factor, and so on.

Examples.—Required the prime factors of 310155.

Dividing by 5, the quotient is 62031. This number is within the range of the tables. But it is not found printed. Therefore 3 is a divisor of it. Dividing by 3, the quotient is 20677. The table gives 23 for the least factor of 20677. Dividing by 23, the quotient is 899.

The table gives 29 for the least factor of 899. Dividing by 29, the quotient is 31, a prime number. Therefore $310155 = 3.5.23.29.31$.

Again, required the divisors of 92881. The table gives 293 for the least divisor. Dividing by it, the quotient is 317. Referring to the tables for 317, a cipher is found in the place of the least divisor, and this signifies that 317 is a prime number.

Therefore $92881 = 293 \times 317$, the product of two primes.

It may be remarked that, to have resolved 92881 into these factors without the aid of the tables by the method of Art. 360, would have involved fifty-nine fruitless trial divisions by prime numbers.

	00	03	06	09	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87
01	0	7	0	17	0	19	0	11	7	37	0	0	13	47	0	7	0	0	11	0	17	0	7	67	19	13	29	0	31	7
07	0	0	0	0	17	11	13	7	29	0	31	0	0	0	7	0	11	0	0	13	0	7	0	0	0	0	37	11	7	0
11	0	0	13	0	7	0	0	0	0	0	0	7	23	0	0	13	17	19	7	0	0	0	11	0	0	7	73	0	13	31
13	0	0	0	11	0	17	7	0	19	0	23	0	0	7	11	0	0	0	0	29	7	59	17	31	0	11	13	7	47	0
17	0	0	0	7	0	37	23	29	0	11	7	31	0	0	0	0	0	7	0	0	11	0	13	0	7	0	0	0	19	23
19	0	11	0	0	23	7	17	13	41	0	0	0	7	0	0	0	61	0	0	7	13	71	0	11	0	73	7	23	0	0
23	0	17	7	13	0	0	0	11	0	7	0	0	0	0	41	0	7	47	11	59	19	0	37	7	31	0	0	0	0	11
29	0	7	17	0	0	11	31	0	7	0	13	0	19	0	0	7	11	23	61	17	0	0	7	13	0	0	0	11	0	7
31	0	0	0	7	0	0	0	0	11	0	7	0	0	0	0	23	0	7	0	11	37	13	19	29	7	17	41	47	0	0
37	0	0	7	0	0	29	11	0	0	7	0	47	0	31	19	13	7	11	0	0	0	0	0	7	0	0	17	79	11	0
41	0	11	0	0	17	23	7	0	0	0	0	13	11	7	0	19	47	53	0	0	7	17	29	11	13	0	0	7	23	0
43	0	7	0	23	11	0	19	0	7	13	17	0	0	0	0	7	29	37	0	0	0	0	7	53	0	19	11	17	0	7
47	0	0	0	0	29	7	0	19	0	41	11	0	7	0	31	0	37	0	13	7	0	11	17	0	0	0	7	0	0	0
49	7	0	11	13	0	0	43	7	31	0	0	17	41	11	7	0	13	19	0	0	23	7	61	0	11	0	47	29	7	13
53	0	0	0	0	7	0	17	0	11	0	43	7	13	59	0	29	23	0	7	11	0	0	0	17	0	7	0	31	79	0
59	0	0	0	7	0	0	11	17	0	31	7	0	0	37	0	47	43	7	53	13	73	0	0	0	7	0	29	41	11	19
61	0	19	0	31	13	7	0	0	23	11	0	0	7	17	0	0	0	13	43	7	11	0	0	0	53	0	7	0	0	0
67	0	0	23	0	7	0	0	11	0	0	0	7	19	0	17	0	31	0	7	73	0	0	59	0	13	7	0	0	0	11
71	0	7	11	0	31	0	0	13	7	17	37	0	0	11	0	7	0	0	0	29	13	23	7	0	11	67	17	0	43	7
73	0	0	0	7	19	11	0	41	0	47	7	0	0	29	0	17	11	7	13	23	0	0	0	19	7	0	0	11	37	31
77	7	13	0	0	0	19	0	7	0	0	17	11	0	41	7	23	0	31	0	53	59	7	11	0	19	0	0	13	7	67
79	0	0	7	11	0	0	0	0	37	7	0	31	13	23	11	19	7	0	0	0	0	0	0	7	29	11	0	0	61	0
83	0	0	0	0	0	0	7	37	13	11	0	17	29	7	0	0	19	71	0	0	7	13	41	0	0	0	0	7	17	0
89	0	0	13	23	0	7	0	11	19	0	0	0	7	0	0	13	0	0	11	7	0	0	0	29	37	0	7	19	13	11
91	7	17	0	0	0	37	31	7	47	0	11	0	0	13	7	0	67	29	17	0	0	7	0	0	23	0	13	0	7	59
97	0	0	17	0	0	0	7	13	11	0	19	43	0	7	0	0	59	0	23	11	7	0	37	0	0	71	53	7	29	19

	01	04	07	10	13	16	19	22	25	28	31	34	37	40	43	46	49	52	55	58	61	64	67	70	73	76	79	82	85	88
01	0	0	0	7	0	0	0	31	41	0	7	19	0	0	11	43	13	7	0	0	0	37	0	0	7	11	0	59	0	13
03	0	13	19	17	0	7	11	0	0	0	29	41	7	0	13	0	0	11	0	7	17	19	0	47	67	0	7	13	11	0
07	0	11	7	19	0	0	0	0	23	7	13	0	11	0	59	17	7	41	0	0	31	43	19	7	0	0	0	29	47	0
09	0	0	0	0	7	0	23	47	13	53	0	7	0	19	31	11	0	0	7	37	41	13	0	43	0	7	11	0	67	23
13	0	7	23	0	13	0	0	0	7	29	11	0	47	0	19	7	17	13	37	0	0	11	7	0	71	23	41	43	0	7
19	7	0	0	0	0	0	19	7	11	0	0	13	0	0	7	31	0	17	0	11	29	7	0	0	13	19	0	0	7	0
21	11	0	7	0	0	0	17	0	0	7	0	11	61	0	29	0	7	23	0	0	0	0	11	7	0	0	89	0	0	0
27	0	7	0	13	0	0	41	17	7	11	53	23	0	0	0	7	13	0	0	0	11	0	7	0	17	29	0	19	0	7
31	0	0	17	0	11	7	0	23	0	19	31	47	7	29	61	11	0	0	0	7	0	59	53	79	0	13	7	0	19	0
33	7	0	0	0	31	23	0	7	17	0	13	0	0	37	7	41	0	0	11	19	0	7	0	13	0	17	0	0	7	11
37	0	19	11	17	7	0	13	0	43	0	0	7	37	11	0	0	0	0	7	13	17	41	0	31	11	7	0	0	0	0
39	0	0	0	0	13	11	7	0	0	17	43	19	0	7	0	0	11	13	29	0	7	47	23	0	41	0	17	7	0	0
43	11	0	0	7	17	31	29	0	0	0	7	11	19	13	43	0	0	7	23	0	0	17	11	0	7	0	13	0	0	37

49	0	0	7	0	19	17	0	13	0	7	47	0	23	0	0	0	7	29	31	0	11	0	17	7	0	0	0	73	83	0
51	0	11	0	0	7	13	0	0	0	0	23	7	11	0	19	0	0	59	7	0	0	0	43	11	0	7	0	37	17	53
57	0	0	0	7	23	0	19	37	0	0	7	0	13	0	0	0	0	7	0	0	47	11	29	0	7	13	73	23	43	17
61	7	0	0	0	0	11	37	7	13	0	29	0	0	31	7	59	11	0	67	0	61	7	0	23	17	47	19	11	7	0
63	0	0	7	0	29	0	13	31	11	7	0	0	53	17	0	0	7	19	0	11	0	23	0	7	37	79	0	0	0	0
67	0	0	13	11	0	0	7	0	17	47	0	0	0	7	11	13	0	23	19	0	7	29	67	37	53	11	31	7	13	0
69	13	7	0	0	37	0	11	0	7	19	0	0	0	13	17	7	0	11	0	0	31	0	7	0	0	0	13	0	11	7
73	0	11	0	29	0	7	0	0	31	13	19	23	7	0	0	0	0	0	0	7	0	0	13	11	73	0	7	0	0	19
79	0	0	19	13	7	23	0	43	0	0	11	7	0	0	29	0	13	0	7	0	37	11	0	0	47	7	79	17	23	13
81	0	13	11	23	0	41	7	0	29	43	0	59	19	7	13	31	17	0	0	0	7	0	0	73	11	0	23	7	0	83
87	11	0	0	0	19	7	0	0	13	0	0	11	7	61	41	43	0	17	37	7	23	13	11	19	83	0	7	0	31	0
91	0	0	7	0	13	19	11	29	0	7	0	0	17	0	0	0	7	11	0	43	41	0	0	7	19	0	61	0	11	17
93	0	17	13	0	7	0	0	0	0	11	31	7	0	0	23	13	0	67	7	71	11	43	0	41	0	7	0	0	13	0
97	0	7	0	0	11	0	0	0	7	0	23	13	0	17	0	7	19	0	29	0	0	73	7	47	13	43	11	0	0	7
99	0	0	17	7	0	0	0	11	23	13	7	0	29	0	53	37	0	7	11	17	0	67	13	31	7	0	19	43	0	11
	02	05	08	11	14	17	20	23	26	29	32	35	38	41	44	47	50	53	56	59	62	65	68	71	74	77	80	83	86	89
03	7	0	11	0	23	13	0	7	19	0	0	31	0	11	7	0	0	0	13	0	0	7	0	0	11	0	53	19	7	29
09	11	0	0	0	0	0	7	0	0	0	0	11	13	7	0	17	0	0	71	19	7	23	11	0	31	13	0	7	0	59
11	0	7	0	11	17	29	0	0	7	41	13	0	37	0	11	7	0	47	31	23	0	17	7	13	0	11	0	0	79	7
17	7	11	19	0	13	17	0	7	0	0	0	0	11	23	7	53	29	13	41	61	0	7	17	11	0	0	0	0	7	37
21	13	0	0	19	7	0	43	11	0	23	0	7	0	13	0	0	0	17	7	31	0	0	19	0	41	7	13	53	37	11
23	0	0	0	0	0	0	7	23	43	37	11	13	0	7	0	0	0	0	0	0	7	11	0	17	13	0	71	7	0	0
27	0	17	0	7	0	11	0	13	37	0	7	0	43	0	19	29	11	7	17	0	13	61	0	0	7	0	23	11	0	79
29	0	23	0	0	0	7	0	17	11	29	0	0	7	0	43	0	47	73	13	7	0	0	0	0	17	59	7	0	0	0
33	0	13	7	11	0	0	19	0	0	7	53	0	0	0	11	0	7	0	43	17	23	47	0	7	0	11	29	13	89	0
39	0	7	0	17	0	37	0	0	7	0	41	0	11	0	23	7	0	19	0	0	17	13	7	11	43	71	0	31	53	7
41	0	0	29	7	11	0	13	0	19	17	7	0	23	41	0	11	71	7	0	13	79	31	0	37	7	0	11	19	0	0
47	13	0	7	31	0	0	23	0	0	7	17	0	0	11	0	47	7	0	0	19	0	0	41	7	11	61	13	17	0	23
51	0	19	23	0	0	17	7	0	11	13	0	53	0	7	0	0	0	0	0	11	7	0	13	0	0	23	83	7	41	0
53	11	7	0	0	0	0	0	13	7	0	0	11	0	0	61	7	31	53	0	0	13	0	7	23	29	0	0	0	17	7
57	0	0	0	13	31	7	11	0	0	0	0	0	7	0	0	67	13	11	0	7	0	79	0	17	0	0	7	61	11	13
59	7	13	0	19	0	0	29	7	0	11	0	0	17	0	7	0	0	23	0	59	11	7	19	0	0	0	0	13	7	17
63	0	0	0	0	7	41	0	17	0	0	13	7	0	23	0	11	61	31	7	67	0	0	0	13	17	7	11	0	0	0
69	0	0	11	7	13	29	0	23	17	0	7	43	53	11	41	19	37	7	0	47	0	0	0	67	7	17	0	0	0	0
71	0	0	13	0	0	7	19	0	0	0	0	0	7	43	17	13	11	41	53	7	0	0	0	71	31	19	7	11	13	0
77	0	0	0	11	7	0	31	0	0	13	29	7	0	0	11	17	0	19	7	43	0	0	13	0	0	7	41	0	0	47
81	0	7	0	0	0	13	0	0	7	11	17	0	0	37	0	7	0	0	13	0	11	0	7	43	0	31	0	17	0	7
83	0	11	0	7	0	0	0	0	0	19	7	0	11	47	0	0	13	7	0	31	61	29	0	11	7	43	59	83	19	13
87	7	0	0	0	0	0	0	7	0	29	19	17	13	53	7	0	0	0	11	0	0	7	71	0	0	13	0	0	7	11
89	17	19	7	29	0	0	0	0	0	7	11	37	0	59	67	0	7	17	0	53	19	11	83	7	0	0	0	0	0	89
93	0	0	19	0	0	11	7	0	0	41	37	0	17	7	0	0	11	0	0	13	7	19	61	0	59	0	0	7	0	17
99	13	0	29	11	0	7	0	0	0	0	0	59	7	13	11	0	0	0	41	7	0	0	0	23	0	11	7	37	0	0

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	361	64	67	70	73	76	79	82	85	88	91	94	397	400	03	06	09	12	15	18	21	24	27	30	33	36	39	42	45	48
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49	37	7	0	0	13	0	137	23	7	53	11	103	0	29	157	7	0	13	0	0	113	11	7	0	67	0	71	0	0	7
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99	53	17	7	23	149	0	13	0	11	7	0	0	0	0	71	0	7	0	17	11	19	0	127	7	0	89	23	31	103	59
	3 62	65	68	71	74	77	80	83	86	89	92	95	3 98	4 01	04	07	10	13	16	19	22	25	28	31	34	37	40	43	46	49
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71	19	0	0	0	7	107	11	0	0	0	173	7	13	17	0	0	67	11	7	19	41	0	43	23	29	7	0	0	11	0
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83	13	0	7	19	0	0	0	131	101	7	163	23	0	11	0	17	7	29	73	0	0	97	19	7	11	0	13	0	0	0
87	131	0	0	41	19	29	7	23	11	13	17	31	0	7	0	0	181	0	0	11	7	37	13	19	0	0	0	7	0	0
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93	0	23	79	13	0	7	11	0	0	0	0	17	7	0	0	19	13	11	173	7	0	191	59	47	23	0	7	103	11	13
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D

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09	79	7	43	139	0	127	61	17	7	0	0	0	67	0	13	7	29	23	53	11	0	101	7	0	17	0	157	13	73	7
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49	13	47	11	0	0	0	7	37	17	59	89	0	29	7	61	131	199	109	0	0	7	0	0	23	11	17	13	7	0	0
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57	7	131	0	11	151	13	0	7	19	0	0	47	0	0	7	17	0	29	13	0	0	7	73	0	41	11	0	19	7	0
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	6 31	34	37	40	43	46	49	52	55	58	61	64	67	70	73	76	79	82	85	88	91	94	6 97	7 00	03	06	09	12	15	18
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49	19	79	0	11	0	7	109	17	29	191	13	0	7	0	11	41	61	0	23	7	0	157	47	13	17	11	7	31	149	0
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47	53	167	0	0	13	43	7	11	193	163	0	17	37	7	79	0	0	13	11	0	7	23	127	29	31	0	0	7	17	11
49	17	7	13	103	0	83	53	43	7	137	11	277	71	0	307	7	0	17	31	23	139	11	7	67	79	0	0	61	13	7
53	0	0	269	19	0	7	31	0	59	0	0	13	7	47	0	23	11	0	53	7	0	0	19	0	13	0	7	11	0	17
59	0	0	0	11	7	13	97	157	0	23	0	7	73	17	11	0	29	43	7	31	0	167	163	0	0	7	0	103	0	61
61	113	109	17	13	263	19	7	23	0	0	29	89	229	7	0	0	13	11	0	17	7	173	0	47	19	0	0	7	11	13
67	0	23	71	17	11	7	0	37	0	0	13	73	7	0	107	11	19	59	0	7	17	29	0	13	23	43	7	89	0	283
71	0	0	7	0	107	0	13	61	89	7	11	0	47	0	31	17	7	19	0	13	23	11	0	7	211	0	0	127	59	43
73	0	0	11	29	7	0	0	0	19	113	163	7	283	11	0	0	0	13	7	0	191	17	277	0	11	7	97	19	0	0
77	13	7	0	0	97	0	79	0	7	19	0	0	113	13	23	7	17	0	307	11	29	0	7	37	89	0	13	31	19	7
79	11	0	0	7	37	17	139	0	0	0	7	11	23	0	29	271	79	7	0	19	0	31	11	0	7	0	0	0	0	0
83	7	19	29	37	0	0	11	7	23	31	0	0	0	0	7	0	239	11	0	0	13	7	109	293	0	0	0	47	7	173
89	0	13	23	0	11	67	7	0	0	0	0	47	19	7	13	11	0	0	17	0	7	113	31	0	271	23	11	7	149	223
91	23	7	89	19	0	0	43	11	7	0	127	61	13	193	0	7	31	0	11	0	307	41	7	23	17	13	53	149	0	7
97	7	0	0	0	0	11	13	7	17	71	0	59	43	0	7	0	11	23	29	13	0	7	0	0	149	17	223	11	7	31

	9 01	04	07	10	13	16	19	22	25	28	31	34	37	40	43	46	49	52	55	58	61	64	67	70	73	76	79	82	85	88
01	11	0	13	17	7	139	29	137	233	0	157	7	0	23	181	13	43	31	7	0	17	0	11	0	0	7	47	283	13	0
03	13	0	0	11	0	47	7	0	0	17	0	23	0	7	11	0	0	0	43	0	7	149	0	0	0	11	13	7	137	29
07	0	0	61	7	17	101	73	19	0	11	7	0	83	0	0	89	0	7	0	149	11	17	13	0	7	0	19	0	0	0
09	251	11	0	0	0	7	0	13	79	0	17	29	7	0	0	37	107	19	149	7	13	229	97	11	31	0	7	17	23	0
13	97	23	7	13	127	17	107	11	71	7	0	109	31	41	37	0	7	0	11	0	223	67	17	7	23	0	179	0	29	11
19	227	7	83	0	53	11	17	0	7	101	13	0	0	149	257	7	11	0	23	0	277	0	7	13	307	31	0	11	0	7
21	0	19	257	7	29	0	0	0	11	0	7	103	17	167	0	0	23	7	59	11	19	13	311	0	7	41	181	0	83	17
27	0	31	7	227	271	59	11	0	67	7	23	0	19	17	0	13	7	11	0	79	97	211	197	7	0	233	0	0	11	37
31	193	11	0	29	0	0	7	149	17	0	0	13	11	7	0	173	59	0	0	61	7	0	0	11	13	17	0	7	37	23
33	173	7	41	0	11	43	149	0	7	13	0	233	67	0	17	7	0	0	83	47	251	73	7	19	131	89	11	23	0	7
37	23	0	31	59	149	7	89	0	37	17	11	223	7	271	29	101	139	131	13	7	0	11	0	23	19	163	7	193	211	0
39	7	0	11	13	241	0	0	7	29	263	0	41	0	11	7	17	13	0	0	239	127	7	0	0	11	251	37	31	7	13
43	109	149	103	181	7	113	0	0	11	227	17	7	13	157	0	31	19	23	7	11	79	0	89	53	311	7	0	17	0	97

49	0	151	0	7	167	37	11	29	19	0	7	17	241	0	0	0	0	7	0	13	0	43	0	107	7	0	41	19	11	0
51	17	29	151	83	13	7	0	0	0	11	0	113	7	163	0	0	0	13	19	7	11	0	31	37	67	0	7	0	139	41
57	89	17	47	23	7	151	0	11	0	0	19	7	29	0	157	103	269	0	7	0	0	0	0	71	13	7	23	0	67	11
61	29	7	11	41	103	71	0	13	7	0	59	19	0	11	127	7	0	0	0	257	13	0	7	31	11	61	0	97	0	7
63	0	61	17	7	211	11	41	257	151	0	7	0	0	0	197	181	11	7	13	17	23	19	0	29	7	127	163	11	0	109
67	7	13	139	19	0	31	0	7	0	0	151	11	41	109	7	137	23	0	227	37	0	7	11	113	0	101	0	13	7	0
69	37	0	7	11	0	29	0	0	0	7	0	151	13	19	11	41	7	47	0	0	17	0	0	7	0	11	313	0	241	0
73	0	0	43	61	0	0	7	53	13	11	23	211	79	7	19	17	73	0	31	0	7	13	29	0	0	0	0	7	0	0
79	31	173	13	0	23	7	19	11	43	131	0	0	7	0	0	13	17	0	11	7	0	0	0	193	0	19	7	23	13	11
81	7	0	23	0	0	17	59	7	0	293	11	0	191	13	7	73	19	151	0	0	0	7	17	0	0	23	13	29	7	61
87	0	41	0	79	0	277	7	13	11	29	0	0	0	7	37	0	43	0	61	11	7	0	0	17	0	0	0	7	311	0
91	0	17	163	7	59	0	67	41	53	19	7	0	71	37	11	23	13	7	17	0	43	47	151	79	7	11	29	227	19	13
93	19	13	0	71	0	7	11	17	0	0	41	0	7	23	13	0	0	11	109	7	29	0	43	151	17	211	7	13	11	0
97	0	11	7	0	0	47	0	0	29	7	13	0	11	73	0	281	7	233	0	17	19	0	0	7	0	151	43	0	0	0
99	0	0	29	0	7	107	197	23	13	0	0	7	97	0	0	11	0	157	7	41	0	13	0	89	173	7	11	0	43	0

	9 02	05	08	11	14	17	20	23	26	29	32	35	38	41	44	47	50	53	56	59	62	65	68	71	74	77	80	83	86	89
03	0	7	0	17	13	0	0	241	7	61	11	0	19	139	67	7	0	13	0	29	17	11	7	0	257	41	23	197	151	7
09	7	29	71	31	17	293	0	7	11	53	83	13	0	0	7	0	0	191	67	11	23	7	131	19	13	199	0	37	7	0
11	11	0	7	179	0	0	101	0	37	7	17	11	0	0	19	53	7	0	23	0	0	103	11	7	29	0	0	17	31	0
17	0	7	197	13	113	41	19	0	7	11	31	17	23	0	263	7	13	0	0	0	11	0	7	0	61	19	0	0	17	7
21	83	131	0	0	11	7	17	19	23	0	73	41	7	0	0	11	0	199	0	7	0	263	0	17	37	13	7	0	0	31
23	7	0	0	293	0	37	23	7	0	43	13	0	17	61	7	0	167	19	11	0	0	7	0	13	0	79	83	0	7	11
27	0	0	11	0	7	29	13	17	0	0	53	7	0	11	0	0	0	0	7	13	41	0	0	0	11	7	61	0	0	0
29	23	0	61	0	13	11	7	127	211	19	0	0	101	7	89	43	11	13	0	0	7	83	37	23	0	0	167	7	19	0
33	11	0	0	7	0	0	0	0	17	199	7	11	103	13	0	61	29	7	0	23	0	37	11	137	7	17	13	107	53	19
39	0	37	7	0	61	199	31	13	0	7	0	89	107	23	0	211	7	0	59	197	11	19	179	7	139	43	17	29	0	0
41	31	11	0	0	7	13	0	107	0	0	0	7	11	47	0	17	101	67	7	37	157	29	113	11	0	7	0	43	0	163
47	0	0	0	7	19	23	83	0	0	41	7	139	13	31	0	0	17	7	101	0	109	11	0	19	7	13	0	0	23	0
51	7	23	47	0	109	11	0	7	13	0	0	17	0	0	7	41	11	97	0	229	29	7	0	0	19	239	71	11	7	53
53	17	83	7	0	0	0	13	0	11	7	0	0	127	0	29	19	7	17	41	11	101	0	23	7	0	67	31	59	47	0
57	43	137	13	11	0	0	7	0	0	0	0	0	17	7	11	13	19	167	23	0	7	0	0	0	41	11	0	7	13	17
59	13	7	43	0	0	89	11	19	7	0	179	0	47	13	59	7	23	11	17	0	0	223	7	0	0	29	13	41	11	7
63	0	11	0	0	0	7	43	0	19	13	0	0	7	17	0	193	0	47	271	7	0	61	13	11	0	59	7	19	0	0
69	19	41	89	13	7	163	23	0	0	31	11	7	37	0	17	97	13	0	7	19	0	11	157	0	29	7	281	0	0	13
71	0	13	11	17	23	0	7	71	0	239	19	137	0	7	13	0	0	283	29	0	7	269	73	0	11	0	101	7	79	19
77	11	53	19	73	17	7	0	0	13	109	37	11	7	41	0	0	31	127	241	7	43	13	11	0	107	0	7	0	101	29
81	0	239	7	19	13	0	11	0	0	7	0	0	269	53	107	0	7	11	163	41	0	0	19	7	43	277	0	131	11	0
83	137	0	13	0	7	17	0	0	0	11	0	7	223	19	0	13	0	0	7	53	11	59	17	157	71	7	43	37	13	31
87	17	7	0	67	11	263	71	0	7	0	0	13	0	97	19	7	0	17	103	0	73	0	7	0	13	0	11	0	29	7
89	0	157	97	7	191	19	17	11	59	13	7	31	0	131	61	0	0	7	11	0	0	0	13	17	7	0	47	0	0	11
93	7	17	11	0	0	13	19	7	0	0	29	173	0	11	7	0	0	0	13	59	0	7	0	83	11	19	233	61	7	0
99	11	0	17	0	0	41	7	0	0	113	79	11	13	7	53	47	61	19	83	17	7	29	11	37	0	13	263	7	229	0

Log Γ(*n*).

n	0	1	2	3	4	5	6	7	8	9
1.00		97497	95001	92512	90030	87555	85087	82627	80173	77727
1.01	9.9975287	72855	70430	68011	65600	63196	60799	58408	56025	53648
1.02	51279	48916	46561	44212	41870	39535	37207	34886	32572	30265
1.03	27964	25671	23384	21104	18831	16564	14305	12052	09806	07567
1.04	05334	03108	00889	$\bar{9}8677$	$\bar{9}6471$	$\bar{9}4273$	$\bar{9}2080$	$\bar{8}9895$	$\bar{8}7716$	$\bar{8}5544$
1.05	9.9883379	81220	79068	76922	74783	72651	70525	68406	66294	64188
1.06	62089	59996	57910	55830	53757	51690	49630	47577	45530	43489
1.07	41469	39428	37407	35392	33384	31382	29387	27398	25415	23449
1.08	21469	19506	17549	15599	13655	11717	09785	07860	05941	04029
1.09	02123	00223	$\bar{9}8329$	$\bar{9}6442$	$\bar{9}4561$	$\bar{9}2686$	$\bar{9}0818$	$\bar{8}8956$	$\bar{8}7100$	$\bar{8}5250$
1.10	9.9783407	81570	79738	77914	76095	74283	72476	70676	68882	67095
1.11	65313	63538	61768	60005	58248	56497	54753	53014	51281	49555
1.12	47834	46120	44411	42709	41013	39323	37638	35960	34288	32622
1.13	30962	29308	27659	26017	24381	22751	21126	19508	17896	16289
1.14	14689	13094	11505	09922	08345	06774	05209	03650	02096	00549
1.15	9.9699007	97471	95941	94417	92898	91386	89879	88378	86883	85393
1.16	83910	82432	80960	79493	78033	76578	75129	73686	72248	70816
1.17	69390	67969	66554	65145	63742	62344	60952	59566	58185	56810
1.18	55440	54076	52718	51366	50019	48677	47341	46011	44687	43368
1.19	42054	40746	39444	38147	36856	35570	34290	33016	31747	30483
1.20	29225	27973	26725	25484	24248	23017	21792	20573	19358	18150
1.21	16946	15748	14556	13369	12188	11011	09841	08675	07515	06361
1.22	05212	04068	02930	01796	00669	$\bar{9}9546$	$\bar{9}8430$	$\bar{9}7318$	$\bar{9}6212$	$\bar{9}5111$
1.23	9.9594015	92925	91840	90760	89685	88616	87553	86494	85441	84393
1.24	83350	82313	81280	80253	79232	78215	77204	76198	75197	74201
1.25	73211	72226	71246	70271	69301	68337	67377	66423	65474	64530
1.26	63592	62658	61730	60806	59888	58975	58067	57165	56267	55374
1.27	54487	53604	52727	51855	50988	50126	49268	48416	47570	46728
1.28	45891	45059	44232	43410	42593	41782	40975	40173	39376	38585
1.29	37798	37016	36239	35467	34700	33938	33181	32439	31682	30940
1.30	30203	29470	28743	28021	27303	26590	25883	25180	24482	23789
1.31	23100	22417	21739	21065	20396	19732	19073	18419	17770	17125
1.32	16485	15850	15220	14595	13975	13359	12748	12142	11540	10944
1.33	10353	09766	09184	08606	08034	07466	06903	06344	05791	05242
1.34	04698	04158	03624	03094	02568	02048	01532	01021	00514	00012
1.35	9.9499515	99023	98535	98052	97573	97100	96630	96166	95706	95251
1.36	94800	94355	93913	93477	93044	92617	92194	91776	91362	90953
1.37	90549	90149	89754	89363	88977	88595	88218	87846	87478	87115
1.38	86756	86402	86052	85707	85366	85030	84698	84371	84049	83731
1.39	83417	83108	82803	82503	82208	81916	81630	81348	81070	80797
1.40	80528	80263	80003	79748	79497	79250	79008	78770	78537	78308
1.41	78084	77864	77648	77437	77230	77027	76829	76636	76446	76261
1.42	76081	75905	75733	75565	75402	75243	75089	74939	74793	74652
1.43	74515	74382	74254	74130	74010	73894	73783	73676	73574	73476
1.44	73382	73292	73207	73125	73049	72976	72908	72844	72784	72728
1.45	72677	72630	72587	72549	72514	72484	72459	72437	72419	72406
1.46	72397	72393	72392	72396	72404	72416	72432	72452	72477	72506
1.47	72539	72576	72617	72662	72712	72766	72824	72886	72952	73022
1.48	73097	73175	73258	73345	73436	73531	73630	73734	73841	73953
1.49	74068	74188	74312	74440	74572	74708	74848	74992	75141	75293

Note.—This table is taken from Vol. II. of Legendre's work, and not from Vol. I., as stated in the Preface: the numbers given in Vol. I. being inaccurate in the seventh decimal place. In Vol II. the values are given to twelve places of decimals. The figure here printed in the seventh place is

n	0	1	2	3	4	5	6	7	8	9
1.50	9.9475449	75610	75774	75943	76116	76292	76473	76658	76847	77040
1.51	77237	77438	77642	77851	78064	78281	78502	78727	78956	79189
1.52	79426	79667	79912	80161	80414	80671	80932	81196	81465	81738
1.53	82015	82295	82580	82868	83161	83457	83758	84062	84370	84682
1.54	84998	85318	85642	85970	86302	86638	86977	87321	87668	88019
1.55	88374	88733	89096	89463	89834	90208	90587	90969	91355	91745
1.56	92139	92537	92938	93344	93753	94166	94583	95004	95429	95857
1.57	96289	96725	97165	97609	98056	98508	98963	99422	99885	$\bar{0}0351$
1.58	9.9500822	01296	01774	02255	02741	03230	03723	04220	04720	05225
1.59	05733	06245	06760	07280	07803	08330	08860	09395	09933	10475
1.60	11020	11569	12122	12679	13240	13804	14372	14943	15519	16098
1.61	16680	17267	17857	18451	19048	19650	20254	20862	21475	22091
1.62	22710	23333	23960	24591	25225	25863	26504	27149	27798	28451
1.63	29107	29767	30430	31097	31767	32442	33120	33801	34486	35175
1.64	35867	36563	37263	37966	38673	39383	40097	40815	41536	42260
1.65	42989	43721	44456	45195	45938	46684	47434	48187	48944	49704
1.66	50468	51236	52007	52782	53560	54342	55127	55916	56708	57504
1.67	58303	59106	59913	60723	61536	62353	63174	63998	64826	65656
1.68	66491	67329	68170	69015	69864	70716	71571	72430	73293	74159
1.69	75028	75901	76777	77657	78540	79427	80317	81211	82108	83008
1.70	83912	84820	85731	86645	87563	88484	89409	90337	91268	92203
1.71	93141	94083	95028	95977	96929	97884	98843	99805	$\bar{0}0771$	$\bar{0}1740$
1.72	9.9602712	03688	04667	05650	06636	07625	08618	09614	10613	11616
1.73	12622	13632	14645	15661	16681	17704	18730	19760	20793	21830
1.74	22869	23912	24959	26009	27062	28118	29178	30241	31308	32377
1.75	33451	34527	35607	36690	37776	38866	39959	41055	42155	43258
1.76	44364	45473	46586	47702	48821	49944	51070	52200	53331	54467
1.77	55606	56749	57894	59043	60195	61350	62509	63671	64836	66004
1.78	67176	68351	69529	70710	71895	73082	74274	75468	76665	77866
1.79	79070	80277	81488	82701	83918	85138	86361	87588	88818	90051
1.80	91287	92526	93768	95014	96263	97515	98770	$\bar{0}0029$	$\bar{0}1291$	$\bar{0}2555$
1.81	9.9703823	05095	06369	07646	08927	10211	11498	12788	14082	15378
1.82	16678	17981	19287	20596	21908	23224	24542	25864	27189	28517
1.83	29848	31182	32520	33860	35204	36551	37900	39254	40610	41969
1.84	43331	44697	46065	47437	48812	50190	51571	52955	54342	55733
1.85	57126	58522	59922	61325	62730	64140	65551	66966	68384	69805
1.86	71230	72657	74087	75521	76957	78397	79839	81285	82734	84186
1.87	85640	87098	88559	90023	91490	92960	94433	95910	97389	98871
1.88	9.9800356	01844	03335	04830	06327	07827	09331	10837	12346	13859
1.89	15374	16893	18414	19939	21466	22996	24530	26066	27606	29148
1.90	30693	32242	33793	35348	36905	38465	40028	41595	43164	44736
1.91	46311	47890	49471	51055	52642	54232	55825	57421	59020	60622
1.92	62226	63834	65445	67058	68675	70294	71917	73542	75170	76802
1.93	78436	80073	81713	83356	85002	86651	88302	89957	91614	93275
1.94	94938	96605	98274	99946	$\bar{0}1621$	$\bar{0}3299$	$\bar{0}4980$	$\bar{0}6663$	$\bar{0}8350$	$\bar{1}0039$
1.95	9.9911732	13427	15125	16826	18530	20237	21947	23659	25375	27093
1.96	28815	30539	32266	33995	35728	37464	39202	40943	42688	44435
1.97	46185	47937	49693	51451	53213	54977	56744	58513	60286	62062
1.98	63840	65621	67405	69192	70982	72774	74570	76368	78169	79972
1.99	81779	83588	85401	87216	89034	90854	92678	94504	96333	98165

the one nearest to the true value whether in excess or defect. This table, and the table of Least Factors, have each been subjected to two complete and independent revisions before finally printing off.

ALGEBRA.

FACTORS.

1 $a^2-b^2 = (a-b)(a+b).$

2 $a^3-b^3 = (a-b)(a^2+ab+b^2).$

3 $a^3+b^3 = (a+b)(a^2-ab+b^2).$

And generally,

4 $a^n-b^n = (a-b)(a^{n-1}+a^{n-2}b+\ldots+b^{n-1})$ always.

5 $a^n-b^n = (a+b)(a^{n-1}-a^{n-2}b+\ldots-b^{n-1})$ if n be even.

6 $a^n+b^n = (a+b)(a^{n-1}-a^{n-2}b+\ldots+b^{n-1})$ if n be odd.

---

7 $(x+a)(x+b) = x^2+(a+b)x+ab.$

8 $(x+a)(x+b)(x+c) = x^3+(a+b+c)x^2 +(bc+ca+ab)x+abc.$

9 $(a+b)^2 = a^2+2ab+b^2.$

10 $(a-b)^2 = a^2-2ab+b^2.$

11 $(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = a^3+b^3+3ab(a+b).$

12 $(a-b)^3 = a^3-3a^2b+3ab^2-b^3 = a^3-b^3-3ab(a-b).$

Generally

$$(a\pm b)^7=a^7\pm 7a^6b+21a^5b^2\pm 35a^4b^3+35a^3b^4\pm 21a^2b^5+7ab^6\pm b^7.$$

Newton's Rule for forming the coefficients—*Multiply any coefficient by the index of the leading quantity, and divide by the number of terms to that place to obtain the coefficient of the term next following.* Thus $21\times 5\div 3$ gives 35, the following coefficient in the example given above. See also (125).

To square a polynomial—*Add to the square of each term twice the product of that term and every term that follows it.*

Thus, $(a+b+c+d)^2$

$$= a^2+2a(b+c+d)+b^2+2b(c+d)+c^2+2cd+d^2.$$

13 $a^4+a^2b^2+b^4 = (a^2+ab+b^2)(a^2-ab+b^2)$.

14 $a^4+b^4 = (a^2+ab\sqrt{2}+b^2)(a^2-ab\sqrt{2}+b^2)$.

15 $\left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2, \quad \left(x+\frac{1}{x}\right)^3 = x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)$

16 $(a+b+c)^2 = a^2+b^2+c^2+2bc+2ca+2ab$.

17 $(a+b+c)^3 = a^3+b^3+c^3+3(b^2c+bc^2+c^2a+ca^2+a^2b+ab^2)+6abc$.

Observe that in an algebraical equation *the sign of any letter may be changed throughout*, and thus a new formula obtained, it being borne in mind that an *even* power of a negative quantity is positive. For example, by changing the sign of c in (16), we obtain

$$(a+b-c)^2 = a^2+b^2+c^2-2bc-2ca+2ab.$$

18 $a^2+b^2-c^2+2ab = (a+b)^2-c^2 = (a+b+c)(a+b-c)$ by (1).

19 $a^2-b^2-c^2+2bc = a^2-(b-c)^2 = (a+b-c)(a-b+c)$.

20 $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-bc-ca-ab)$.

21 $bc^2+b^2c+ca^2+c^2a+ab^2+a^2b+a^3+b^3+c^3 = (a+b+c)(a^2+b^2+c^2)$.

22 $bc^2+b^2c+ca^2+c^2a+ab^2+a^2b+3abc = (a+b+c)(bc+ca+ab)$.

23 $bc^2+b^2c+ca^2+c^2a+ab^2+a^2b+2abc=(b+c)(c+a)(a+b)$

24 $bc^2+b^2c+ca^2+c^2a+ab^2+a^2b-2abc-a^3-b^3-c^3 = (b+c-a)(c+a-b)(a+b-c)$.

25 $bc^2-b^2c+ca^2-c^2a+ab^2-a^2b = (b-c)(c-a)(a-b)$.

26 $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

27 $x^3+2x^2y+2xy^2+y^3 = (x+y)(x^2+xy+y^2)$.

Generally for the division of $(x+y)^n-(x^n+y^n)$ by x^2+xy+y^2 see (545).

MULTIPLICATION AND DIVISION,

BY THE METHOD OF DETACHED COEFFICIENTS.

28 Ex. 1: $(a^4-3a^2b^2+2ab^3+b^4)\times(a^3-2ab^2-2b^3)$.

$$\begin{array}{l}
1+0-3+2+1 \\
1+0-2-2 \\
\hline
1+0-3+2+1 \\
-2-0+6-4-2 \\
-2-0+6-4-2 \\
\hline
1+0-5+0+7+2-6-2
\end{array}$$

Result $a^7-5a^5b^2+7a^3b^4+2a^2b^5-6ab^6-2b^7$.

Ex. 2: $(x^7-5x^5+7x^3+2x^2-6x-2)\div(x^4-3x^2+2x+1)$.

$$\begin{array}{l}
1+0-3+2+1)\ 1+0-5+0+7+2-6-2\ (1+0-2-2 \\
\underline{-1-0+3-2-1} \\
0-2-2+6+2-6 \\
\underline{+2+0-6+4+2} \\
-2+0+6-4-2 \\
\underline{+2+0-6+4+2}
\end{array}$$

Result x^3-2x-2.

Synthetic Division.

Ex. 3: Employing the last example, the work stands thus,

$$\begin{array}{r|l}
 & 1+0-5+0+7+2-6-2 \\
-0 & 0+0+0+0 \\
+3 & +3+0-6-6 \\
-2 & -2+0+4+4 \\
-1 & -1+0+2+2 \\
\hline
 & 1+0-2-2
\end{array}$$

Result x^3-2x-2. [See also (248).

Note that, in all operations with detached coefficients, the result must be written out in successive powers of the quantity which stood in its successive powers in the original expression.

INDICES.

29 Multiplication: $a^{\frac{1}{3}} \times a^{\frac{1}{2}} = a^{\frac{1}{3}+\frac{1}{2}} = a^{\frac{5}{6}}$, or $\sqrt[6]{a^5}$;

$$a^{\frac{1}{m}} \times a^{\frac{1}{n}} = a^{\frac{1}{m}+\frac{1}{n}} = a^{\frac{m+n}{mn}}, \text{ or } \sqrt[mn]{a^{m+n}}.$$

Division: $a^{\frac{2}{3}} \div a^{\frac{1}{2}} = a^{\frac{2}{3}-\frac{1}{2}} = a^{\frac{1}{6}}$, or $\sqrt[6]{a}$;

$$a^{\frac{1}{n}} \div a^{\frac{1}{m}} = a^{\frac{1}{n}-\frac{1}{m}} = a^{\frac{m-n}{mn}}, \text{ or } \sqrt[mn]{a^{m-n}}.$$

Involution: $(a^{\frac{2}{3}})^{\frac{1}{2}} = a^{\frac{2}{3}\times\frac{1}{2}} = a^{\frac{1}{3}}$, or $\sqrt[3]{a}$.

Evolution: $\sqrt[7]{a^{\frac{2}{3}}} = a^{\frac{2}{3}\times\frac{1}{7}} = a^{\frac{2}{21}}$, or $\sqrt[21]{a^2}$.

$$a^{-n} = \frac{1}{a^n}, \quad a^0 = 1.$$

HIGHEST COMMON FACTOR.

30 Rule.—To find the highest common factor of two expressions—*Divide the one which is of the highest dimension by the other, rejecting first any factor of either expression which is not also a factor of the other. Operate in the same manner upon the remainder and the divisor, and continue the process until there is no remainder. The last divisor will be the highest common factor required.*

31 Example.—To find the H. C. F. of

$3x^5-10x^3+15x+8$ and $x^5-2x^4-6x^3+4x^2+13x+6$.

	1− 2− 6+ 4+13+ 6	3+0−10+ 0+15+ 8	3
	3	−3+6+18−12−39−18	
1	3− 6−18+12+39+18	2) 6+ 8−12−24−10	
	−3− 4+ 6+12+ 5	3+ 4− 6−12− 5	
	2) −10−12+24+44+18	−3− 9− 9− 3	
	− 5− 6+12+22+ 9	− 5−15−15− 5	
	3	+ 5+15+15+ 5	
5	−15−18+36+66+27		
	+15+20−30−60−25		
	2) 2+ 6+ 6+ 2		
	1+ 3+ 3+ 1		

Result H. C. F. $= x^3+3x^2+3x+1$.

32 Otherwise.—To form the H. C. F. of two or more algebraical expressions—*Separate the expressions into their simplest factors. The H. C. F. will be the product of the factors common to all the expressions, taken in the lowest powers that occur.*

LOWEST COMMON MULTIPLE.

33 *The L. C. M. of two quantities is equal to their product divided by the H. C. F.*

34 Otherwise.—To form the L. C. M. of two or more algebraical expressions—*Separate them into their simplest factors. The L. C. M. will be the product of all the factors that occur, taken in the highest powers that occur.*

Example.—The H. C. F. of $a^2(b-x)^5c^7d$ and $a^3(b-x)^2c^4e$ is $a^2(b-x)^2c^4$; and the L. C. M. is $a^3(b-x)^5c^7de$.

EVOLUTION.

To extract the square root of

$$a^2-\frac{3a\sqrt{a}}{2}-\frac{3\sqrt{a}}{2}+\frac{41a}{16}+1.$$

Arranging according to powers of a, and reducing to one denominator, the expression becomes $\dfrac{16a^2-24a^{\frac{3}{2}}+41a-24a^{\frac{1}{2}}+16}{16}$.

35 Detaching the coefficients, the work is as follows:—

$$\begin{array}{l|l}
 & 16-24+41-24+16\;(4-3+4 \\
 & 16 \\ \hline
8-3 & -24+41 \\
\;\;-3 & \;\;24-9 \\ \hline
8-6+4 & \;\;32-24+16 \\
 & -32+24-16 \\ \hline
\end{array}$$

Result $\dfrac{4a-3a^{\frac{1}{2}}+4}{4}=a-\frac{3}{4}\sqrt{a}+1.$

	9 00	03	06	09	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87
01	0	73	7	0	11	37	0	31	0	7	0	13	0	0	0	11	7	0	0	0	0	23	0	7	13	0	11	0	19	89
07	0	7	11	0	223	13	0	0	7	0	17	0	0	11	0	7	113	0	13	0	19	193	7	0	11	281	47	17	0	7
11	0	13	19	0	197	7	0	0	11	83	281	23	7	0	13	29	0	0	73	7	67	19	17	0	41	0	7	13	0	0
13	7	0	31	229	53	0	0	7	0	23	47	11	13	0	7	0	59	227	0	0	0	7	11	199	0	13	0	41	7	0
17	0	37	0	0	7	23	11	251	13	0	191	7	179	19	71	47	53	11	7	0	0	13	79	17	67	7	29	59	11	0
19	0	181	0	23	19	71	7	0	0	11	167	0	17	7	0	31	0	73	0	13	7	61	53	19	191	113	23	7	0	17
23	0	41	13	7	11	19	0	17	29	0	7	0	251	0	59	11	0	7	37	0	131	0	23	103	7	0	11	0	13	269
29	197	59	7	79	0	0	229	181	17	7	41	0	0	11	0	0	7	251	0	29	109	0	13	7	11	17	0	0	0	0
31	0	103	0	0	7	11	131	13	0	47	31	7	109	29	17	0	11	0	7	0	13	0	71	0	0	7	19	11	257	0
37	179	13	233	7	0	239	0	199	23	0	7	0	0	0	11	17	0	7	19	0	137	0	41	31	7	11	227	13	173	0
41	7	61	0	211	23	0	0	7	97	11	13	31	29	0	7	0	0	89	0	19	11	7	241	13	0	103	0	17	7	293
43	127	11	7	199	0	31	29	0	13	7	19	269	11	37	73	0	7	0	0	67	0	13	0	7	47	23	0	0	0	19
47	53	167	0	0	13	43	7	11	193	163	0	17	37	7	79	0	0	13	11	0	7	23	127	29	31	0	0	7	17	11
49	17	7	13	103	0	83	53	43	7	137	11	277	71	0	307	7	0	17	31	23	139	11	7	67	79	0	0	61	13	7
53	0	0	269	19	0	7	31	0	59	0	0	13	7	47	0	23	11	0	53	7	0	0	19	0	13	0	7	11	0	17
59	0	0	0	11	7	13	97	157	0	23	0	7	73	17	11	0	29	43	7	31	0	167	163	0	0	7	0	103	0	61
61	113	109	17	13	263	19	7	23	0	0	29	89	229	7	0	0	13	11	0	17	7	173	0	47	19	0	0	7	11	13
67	0	23	71	17	11	7	0	37	0	0	13	73	7	0	107	11	19	59	0	7	17	29	0	13	23	43	7	89	0	283
71	0	0	7	0	107	0	13	61	89	7	11	0	47	0	31	17	7	19	0	13	23	11	0	7	211	0	0	127	59	43
73	0	0	11	29	7	0	0	0	19	113	163	7	283	11	0	0	0	13	7	0	191	17	277	0	11	7	97	19	0	0
77	13	7	0	0	97	0	79	0	7	19	0	0	113	13	23	7	17	0	307	11	29	0	7	37	89	0	13	31	19	7
79	11	0	0	7	37	17	139	0	0	0	7	11	23	0	29	271	79	7	0	19	0	31	11	0	7	0	0	0	0	0
83	7	19	29	37	0	0	11	7	23	31	0	0	0	0	7	0	239	11	0	0	13	7	109	293	0	0	0	47	7	173
89	0	13	23	0	11	67	7	0	0	0	0	47	19	7	13	11	0	0	17	0	7	113	31	0	271	23	11	7	149	223
91	23	7	89	19	0	0	43	11	7	0	127	61	13	193	0	7	31	0	11	0	307	41	7	23	17	13	53	149	0	7
97	7	0	0	0	0	11	13	7	17	71	0	59	43	0	7	0	11	23	29	13	0	7	0	0	149	17	223	11	7	31

	9 01	04	07	10	13	16	19	22	25	28	31	34	37	40	43	46	49	52	55	58	61	64	67	70	73	76	79	82	85	88
01	11	0	13	17	7	139	29	137	233	0	157	7	0	23	181	13	43	31	7	0	17	0	11	0	0	7	47	283	13	0
03	13	0	0	11	0	47	7	0	0	17	0	23	0	7	11	0	0	0	43	0	7	149	0	0	0	11	13	7	137	29
07	0	0	61	7	17	101	73	19	0	11	7	0	83	0	0	89	0	7	0	149	11	17	13	0	7	0	19	0	0	0
09	251	11	0	0	0	7	0	13	79	0	17	29	7	0	0	37	107	19	149	7	13	229	97	11	31	0	7	17	23	0
13	97	23	7	13	127	17	107	11	71	7	0	109	31	41	37	0	7	0	11	0	223	67	17	7	23	0	179	0	29	11
19	227	7	83	0	53	11	17	0	7	101	13	0	0	149	257	7	11	0	23	0	277	0	7	13	307	31	0	11	0	7
21	0	19	257	7	29	0	0	0	11	0	7	103	17	167	0	0	23	7	59	11	19	13	311	0	7	41	181	0	83	17
27	0	31	7	227	271	59	11	0	67	7	23	0	19	17	0	13	7	11	0	79	97	211	197	7	0	233	0	0	11	37
31	193	11	0	29	0	0	7	149	17	0	0	13	11	7	0	173	59	0	0	61	7	0	0	11	13	17	0	7	37	23
33	173	7	41	0	11	43	149	0	7	13	0	233	67	0	17	7	0	0	83	47	251	73	7	19	131	89	11	23	0	7
37	23	0	31	59	149	7	89	0	37	17	11	223	7	271	29	101	139	131	13	7	0	11	0	23	19	163	7	193	211	0
39	7	0	11	13	241	0	0	7	29	263	0	41	0	11	7	17	13	0	0	239	127	7	0	0	11	251	37	31	7	13
43	109	149	103	181	7	113	0	0	11	227	17	7	13	157	0	31	19	23	7	11	79	0	89	53	311	7	0	17	0	97

42 $$(x+y)^2 = (x-y)^2+4xy.$$

43 $$(x-y)^2 = (x+y)^2-4xy.$$

44 EXAMPLES.

$$\frac{2\sqrt{a^2-b^2}+\sqrt{c^2-x^2}}{2\sqrt{a^2-b^2}-\sqrt{c^2-x^2}} = \frac{3\sqrt{a^2-b^2}+\sqrt{c^2-d^2}}{3\sqrt{a^2-b^2}-\sqrt{c^2-d^2}}$$

$$\frac{\sqrt{c^2-x^2}}{2\sqrt{a^2-b^2}} = \frac{\sqrt{c^2-d^2}}{3\sqrt{a^2-b^2}} \qquad [38$$

$$9(c^2-x^2) = 4(c^2-d^2)$$

$$x = \sqrt{5c^2+4d^2}.$$

To simplify a compound fraction, as

$$\frac{\dfrac{1}{a^2-ab+b^2}+\dfrac{1}{a^2+ab+b^2}}{\dfrac{1}{a^2-ab+b^2}-\dfrac{1}{a^2+ab+b^2}}$$

multiply the numerator and denominator by the L. C. M. of all the smaller denominators.

Result $$\frac{(a^2+ab+b^2)+(a^2-ab+b^2)}{(a^2+ab+b^2)-(a^2-ab+b^2)} = \frac{a^2+b^2}{ab}.$$

QUADRATIC EQUATIONS.

45 If $ax^2+bx+c=0$, $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

46 If $ax^2+2bx+c=0$; that is, if the coefficient of x be an even number, $x = \dfrac{-b\pm\sqrt{b^2-ac}}{a}$.

47 ***Method of solution without the formula.***

Ex.: $$2x^2-7x+3=0.$$

Divide by 2, $$x^2-\frac{7}{2}x+\frac{3}{2}=0$$

Complete the square, $x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = \frac{49}{16} - \frac{3}{2} = \frac{25}{16}$

Take square root, $x - \frac{7}{4} = \pm\frac{5}{4}$

$$x = \frac{7 \pm 5}{4} = 3 \text{ or } \frac{1}{2}.$$

48 Rule for "completing the square" of an expression like $x^2 - \frac{7}{2}x$, *Add the square of half the coefficient of x.*

49 The solution of the foregoing equation, employing formula (45), is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{49 - 24}}{4} = \frac{7 \pm 5}{4} = 3 \text{ or } \frac{1}{2}.$$

THEORY OF QUADRATIC EXPRESSIONS.

If α, β be the roots of the equation $ax^2 + bx + c = 0$, then

50 $$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

51 Sum of roots $\alpha + \beta = -\frac{b}{a}$.

52 Product of roots $\alpha\beta = \frac{c}{a}$.

Condition for the existence of equal roots—

53 $b^2 - 4ac$ must vanish.

54 The solution of equations in one unknown quantity may sometimes be simplified by changing the quantity sought.

Ex. (1): $$2x + \frac{3x-1}{3x+1} + \frac{18x+6}{6x^2+5x-1} = 14 \quad \text{.................... (1).}$$

$$\frac{6x^2+5x-1}{3x+1} + \frac{6(3x+1)}{6x^2+5x-1} = 14$$

Put $$y = \frac{6x^2+5x-1}{3x+1} \quad \text{.............................. (2)}$$

thus $$y+\frac{6}{y}=14$$
$$y^2-14y+6=0$$
y having been determined from this quadratic, x is afterwards found from (2).

---

55 Ex. 2: $$x^2+\frac{1}{x^2}+x+\frac{1}{x}=4.$$
$$\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=6$$
Put $x+\frac{1}{x}=y$, and solve the quadratic in y.

---

56 Ex. 3: $$x^2+x+3\sqrt{2x^2+x+2}=\frac{x}{2}+1.$$
$$2x^2+x+3\sqrt{2x^2+x+2}=2$$
$$2x^2+x+2+3\sqrt{2x^2+x+2}=4$$
Put $\sqrt{2x^2+x+2}=y$, and solve the quadratic
$$y^2+3y=4.$$

---

57 Ex. 4: $$\sqrt[3]{x^n}+\frac{2}{3\sqrt[3]{x^n}}=\frac{16}{3}x^{-n}.$$
$$x^{\frac{4n}{3}}+\frac{2}{3}x^{\frac{2n}{3}}=\frac{16}{3}$$
A quadratic in $y=x^{\frac{2n}{3}}$.

---

58 *To find Maxima and Minima values by means of a Quadratic Equation.*

Ex.—Given $y=3x^2+6x+7$,
to find what value of x will make y a maximum or minimum.

Solve the quadratic equation
$$3x^2+6x+7-y=0$$
Thus $$x=\frac{-3\pm\sqrt{3y-12}}{3} \qquad [46$$

In order that x may be a real quantity, we must have $3y$ not less than 12; therefore 4 is a minimum value of y, and the value of x which makes y a minimum is -1.

SIMULTANEOUS EQUATIONS.

General solution with two unknown quantities.

Given

59 $$\left.\begin{array}{l} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{array}\right\},\quad x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1},\quad y=\frac{c_1a_2-c_2a_1}{b_1a_2-b_2a_1}.$$

General solution with three unknown quantities.

60 Given $$\left.\begin{array}{l} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{array}\right\}$$

$$x=\frac{d_1(b_2c_3-b_3c_2)+d_2(b_3c_1-b_1c_3)+d_3(b_1c_2-b_2c_1)}{a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)}$$

and symmetrical forms for y and z.

Methods of solving simultaneous equations between two unknown quantities x and y.

61 I. ***By substitution.****—Find one unknown in terms of the other from one of the two equations, and substitute this value in the remaining equation. Then solve the resulting equation.*

Ex.: $$\left.\begin{array}{l} x+5y=23 \;\ldots\ldots\; (1) \\ 7y=28 \;\ldots\ldots\; (2) \end{array}\right\}.$$

From (2), $y=4$. Substitute in (1); thus

$$x+20=23,\quad x=3.$$

62 II. ***By the method of Multipliers.***

Ex.: $$\left.\begin{array}{l} 3x+5y=36 \;\ldots\ldots\; (1) \\ 2x-3y=\;\;5 \;\ldots\ldots\; (2) \end{array}\right\}.$$

Eliminate x by multiplying eq. (1) by 2, and (2) by 3; thus

$$\begin{aligned} 6x+10y &= 72 \\ 6x-\;\,9y &= 15 \\ 19y &= 57, \text{ by subtraction,} \\ y &= \;3 \\ \therefore\quad x &= \;7, \text{ by substitution in eq. (2).} \end{aligned}$$

63 III. *By changing the quantities sought.*

Ex. 1:
$$\left.\begin{aligned} x-y &= 2 \;\ldots\ldots\; (1) \\ x^2-y^2+x+y &= 30 \;\ldots\ldots\; (2) \end{aligned}\right\}.$$

Let $x+y = u, \quad x-y = v.$

Substitute these values in (1) and (2),

$$\left.\begin{aligned} v &= 2 \\ uv+u &= 30 \end{aligned}\right\}$$

$$\therefore \quad 2u+u = 30$$
$$u = 10$$
$$\therefore \quad x+y = 10$$
$$x-y = 2$$

From which $x = 6$ and $y = 4$.

---

64 Ex. 2:
$$\left.\begin{aligned} 2\frac{x+y}{x-y} + 10\frac{x-y}{x+y} &= 9 \;\ldots\ldots\; (1) \\ x^2+7y^2 &= 64 \;\ldots\ldots\; (2) \end{aligned}\right\}.$$

Substitute z for $\frac{x+y}{x-y}$ in (1);

$$\therefore \quad 2z + \frac{10}{z} = 9$$
$$2z^2-9z+10 = 0$$

From which $z = \frac{5}{2}$ or 2

$$\frac{x+y}{x-y} = 2 \text{ or } \frac{5}{2}$$

From which $x = 3y$ or $\frac{7}{3}y$

Substitute in (2); thus $y = 2$ and $x = 6$

or $y = \frac{6}{\sqrt{7}}$ and $x = 2\sqrt{7}$.

---

65 Ex. 3:
$$\left.\begin{aligned} 3x+5y &= xy \;\ldots\ldots\; (1) \\ 2x+7y &= 3xy \;\ldots\ldots\; (2) \end{aligned}\right\}.$$

Divide each equation by xy,

$$\left.\begin{aligned} \frac{3}{y} + \frac{5}{x} &= 1 \;\ldots\ldots\; (3) \\ \frac{2}{y} + \frac{7}{x} &= 3 \;\ldots\ldots\; (4) \end{aligned}\right\}$$

Multiply (3) by 2, and (4) by 3, and by subtraction y is eliminated.

66 IV. ***By substituting $y=tx$, when the equations are homogeneous in the terms which contain x and y.***

Ex. 1: $$\left.\begin{aligned} 52x^2+7xy &= 5y^2 \;\ldots\ldots\; (1) \\ 5x-3y &= 17 \;\ldots\ldots\; (2) \end{aligned}\right\}.$$

From (1), and, from (2), $$\left.\begin{aligned} 52x^2+7tx^2 &= 5t^2x \ldots\ldots (3) \\ 5x-3tx &= 17 \;\ldots\ldots\; (4) \end{aligned}\right\}$$

(3) gives $52+7t = 5t^2$

a quadratic equation from which t must be found, and its value substituted in (4).

x is thus determined; and then y from $y = tx$.

---

67 Ex. 2: $$\left.\begin{aligned} 2x^2+xy+3y^2 &= 16 \;\ldots\ldots\; (1) \\ 3y-2x &= \;\;4 \;\ldots\ldots\; (2) \end{aligned}\right\}.$$

From (1), by putting $y = tx$,

$$\left.\begin{aligned} x^2(2+t+3t^2) &= 16 \;\ldots\ldots\; (3) \\ x(3t-2) &= \;\;4 \;\ldots\ldots\; (4) \end{aligned}\right\}$$

From (2),

squaring, $x^2(9t^2-12t+4) = 16$

$\therefore \quad 9t^2-12t+4 = 2+t+3t^2$

a quadratic equation for t.

t being found from this, equation (4) will determine x; and finally $y=tx$.

RATIO AND PROPORTION.

---

68 If $a : b :: c : d$; then $ad = bc$, and $\frac{a}{b} = \frac{c}{d}$;

$$\frac{a+b}{b} = \frac{c+d}{d}; \quad \frac{a-b}{b} = \frac{c-d}{d}; \quad \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

---

69 If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} =$ &c.; then $\frac{a}{b} = \frac{a+c+e+\&c.}{b+d+f+\&c.}$.

General theorem.

70 If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} =$ &c. $= k$ say; then

$$k = \left\{\frac{pa^n+qc^n+re^n+\&c.}{pb^n+qd^n+rf^n+\&c.}\right\}^{\frac{1}{n}}$$

where p, q, r, &c. are any quantities whatever. Proved as in (71).

71 RULE.—To verify any equation between such proportional quantities—*Substitute for a, c, e, &c. their equivalents kb, kd, kf, &c. respectively, in the given equation.*

Ex.—If $a : b :: c : d$, to show that

$$\frac{\sqrt{a-b}}{\sqrt{c-d}} = \frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}-\sqrt{d}}.$$

Put kb for a, and kd for c; thus

$$\frac{\sqrt{a-b}}{\sqrt{c-d}} = \frac{\sqrt{kb-b}}{\sqrt{kd-d}} = \frac{\sqrt{b}\sqrt{k-1}}{\sqrt{d}\sqrt{k-1}} = \frac{\sqrt{b}}{\sqrt{d}}$$

also $$\frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}-\sqrt{d}} = \frac{\sqrt{kb}-\sqrt{b}}{\sqrt{kd}-\sqrt{d}} = \frac{\sqrt{b}\,(\sqrt{k}-1)}{\sqrt{d}\,(\sqrt{k}-1)} = \frac{\sqrt{b}}{\sqrt{d}}$$

Identical results being obtained, the proposed equation must be true.

---

72 If $\boldsymbol{a : b : c : d : e}$ &c., forming a continued proportion.
Then $\boldsymbol{a : c :: a^2 : b^2}$ the duplicate ratio of $\boldsymbol{a : b}$,
$\boldsymbol{a : d :: a^3 : b^3}$ the triplicate ratio of $\boldsymbol{a : b}$, and so on.
Also $\sqrt{a} : \sqrt{b}$ is the subduplicate ratio of $a : b$,
$a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is the sesquiplicate ratio of $a : b$.

---

73 The fraction $\frac{a}{b}$ is made to approach nearer to unity in value, by adding the same quantity to the numerator and denominator. Thus

$$\frac{a+x}{b+x} \text{ is nearer to 1 than } \frac{a}{b} \text{ is.}$$

---

74 DEF.—The ratio compounded of the ratios $a : b$ and $c : d$ is the ratio $ac : bd$.

75 If $a : b :: c : d$, and $a' : b' :: c' : d'$; then, by compounding ratios, $aa' : bb' :: cc' : dd'$.

VARIATION.

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76 If $a \propto c$ and $b \propto c$, then $(a \pm b) \propto c$ and $\sqrt{ab} \propto c$.

77 If $\left.\begin{array}{l} a \propto b \\ \text{and } c \propto d \end{array}\right\}$, then $ac \propto bd$ and $\frac{a}{c} \propto \frac{b}{d}$.

78 If $a \propto b$, we may assume $a = mb$, where m is some constant.

ARITHMETICAL PROGRESSION.

General form of a series in A. P.

79 $a,\ a+d,\ a+2d,\ a+3d,\ \ldots\ldots\ a+(n-1)\,d.$

$a =$ first term,
$d =$ common difference,
$l =$ last of n terms,
$s =$ sum of n terms; then

80 $$l = a+(n-1)\,d.$$

81 $$s = (a+l)\,\frac{n}{2}.$$

82 $$s = \{2a+(n-1)\,d\}\,\frac{n}{2}.$$

Obtained by writing (79) in reversed order, and adding both series together.

GEOMETRICAL PROGRESSION.

General form of a series in G. P.

83 $a,\ ar,\ ar^2,\ ar^3,\ \ldots\ldots\ ar^{n-1}.$

$a =$ first term,
$r =$ common ratio,
$l =$ last of n terms,
$s =$ sum of n terms; then

84 $$l = ar^{n-1}.$$

85 $$s = a\,\frac{r^n-1}{r-1} \quad \text{or} \quad a\,\frac{1-r^n}{1-r}.$$

If r be less than 1, and n be infinite,

86 $$s = \frac{a}{1-r}, \quad \text{since} \quad r^n = 0.$$

(85) is obtained by multiplying (83) by r, and subtracting one series from the other.

HARMONICAL PROGRESSION.

87 a, b, c, d, &c. are in Harm. Prog. when the reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c. are in Arith. Prog.

88 Or when $a : b :: a-b : b-c$ is the relation subsisting between any three consecutive terms.

89 n^{th} term of the series $= \dfrac{ab}{(n-1)a-(n-2)b}$. [87, 80.

90 Approximate sum of n terms of the Harm. Prog. $\frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}$, &c., when d is small compared with a,

$$= \frac{(a+d)^n - a^n}{d\,(a+d)^n}.$$

By taking instead the G. P. $\dfrac{1}{a+d} + \dfrac{1}{(a+d)^2} + \dfrac{1}{(a+d)^3} + \ldots$

91 Arithmetic mean between a and $b = \dfrac{a+b}{2}$.

92 Geometric do. $= \sqrt{ab}$.

93 Harmonic do. $= \dfrac{2ab}{a+b}$.

The three means are in continued proportion.

PERMUTATIONS AND COMBINATIONS.

94 The number of permutations of n things taken *all* at a time $= n(n-1)(n-2)\ldots 3.2.1 \equiv \underline{|n}$ or $n^{(n)}$.

PROOF BY INDUCTION.—Assume the formula to be true for n things. Now take $n+1$ things. After each of these the remaining n things may be arranged in $\underline{|n}$ ways, making in all $n \times \underline{|n}$ (that is, $\underline{|n+1}$) permutations of $n+1$ things; therefore, &c. See also (233) for the mode of proof by Induction.

95 The number of permutations of n things taken r at a time is denoted by $P(n, r)$.

$$P(n, r) = n(n-1)(n-2)\dots(n-r+1) \equiv n^{(r)}.$$

By (94); for $(n-r)$ things are left out of each permutation; therefore $P(n, r) = \lfloor n \div \lfloor n-r$.

Observe that $r =$ the number of factors.

---

96 The number of combinations of n things taken r at a time is denoted by $C(n, r)$.

$$C(n, r) = \frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r} \equiv \frac{n^{(r)}}{\lfloor r}$$

$$= \frac{\lfloor n}{\lfloor r \lfloor n-r} = C(n, n-r).$$

For every combination of r things admits of $\lfloor r$ permutations; therefore $C(n, r) = P(n, r) \div \lfloor r$.

97 $C(n, r)$ is greatest when $r = \frac{1}{2}n$ or $\frac{1}{2}(n \pm 1)$, according as n is even or odd.

---

98 The number of homogeneous products of r dimensions of n things is denoted by $H(n, r)$.

$$H(n, r) = \frac{n(n+1)(n+2)\dots(n+r-1)}{1.2\dots r} \equiv \frac{(n+r-1)^{(r)}}{\lfloor r}.$$

When r is $> n$, this reduces to

99 $$\frac{(r+1)(r+2)\dots(n+r-1)}{\lfloor r-1}.$$

Proof.—$H(n, r)$ is equal to the number of terms in the product of the expansions by the Bin. Th. of the n expressions $(1-ax)^{-1}$, $(1-bx)^{-1}$, $(1-cx)^{-1}$, &c.

Put $a = b = c =$ &c. $= 1$. The number will be the coefficient of x^r in $(1-x)^{-n}$. (128, 129.)

100 The number of permutations of n things taken all together, when a of them are alike, b of them alike, c alike, &c.

$$= \frac{\lfloor n}{\lfloor a \, \lfloor b \, \lfloor c \ldots \&c.}.$$

For if the a things were all different, they would form $\lfloor a$ permutations where there is now but one. So of b, c, &c.

---

101 The number of combinations of n things r at a time, in which any p of them will always be found, is

$$= C(n-p, r-p).$$

For if the p things be set on one side, we have to add to them $r-p$ things taken from the remaining $n-p$ things in every possible way.

---

102 THEOREM: $C(n-1, r-1)+C(n-1, r) = C(n, r)$.

PROOF BY INDUCTION; or as follows: Put one out of n letters aside; there are $C(n-1, r)$ combinations of the remaining $n-1$ letters r at a time. To complete the total $C(n, r)$, we must place with the excluded letter all the combinations of the remaining $n-1$ letters $r-1$ at a time.

---

103 If there be one set of P things, another of Q things, another of R things, and so on; the number of combinations formed by taking one out of each set is $= PQR \ldots$ &c., the product of the numbers in the several sets.

For one of the P things will form Q combinations with the Q things. A second of the P things will form Q more combinations; and so on. In all, PQ combinations of two things. Similarly there will be PQR combinations of three things; and so on. This principle is very important.

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104 On the same principle, if p, q, r, &c. things be taken out of each set respectively, the number of combinations will be the product of the numbers of the separate combinations;

that is, $= C(Pp) \,.\, C(Qq) \,.\, C(Rr) \ldots$ &c.

105 The number of combinations of n things taken m at a time, when p of the n things are alike, q of them alike, r of them alike, &c., will be the sum of all the combinations of each possible form of m dimensions, and this is equal to the coefficient of x^m in the expansion of

$$(1+x+x^2+\ldots+x^p)\,(1+x+x^2+\ldots+x^q)\,(1+x+x^2+\ldots+x^r)\ldots.$$

106 The total number of possible combinations under the same circumstances, when the n things are taken in all ways, 1, 2, 3 … n at a time

$$= (p+1)\,(q+1)\,(r+1)\ldots-1.$$

107 The number of permutations when they are taken in all possible ways will be equal to the product of $\lfloor m$ and the coefficient of x^m in the expansion of

$$\left\{1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots+\frac{x^p}{\lfloor p}\right\}\left\{1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots+\frac{x^q}{\lfloor q}\right\}\ldots$$

……&c.

SURDS.

108 To reduce $\sqrt[3]{2808}$. Decompose the number into its prime factors by (196); thus,

$$\sqrt[3]{2808} = \sqrt[3]{2^3 . 3^3 . 13} = 6\sqrt[3]{13}$$

$$\sqrt[3]{a^{15}b^{10}c^8} = a^{\frac{15}{3}}\,b^{\frac{10}{3}}\,c^{\frac{8}{3}} = a^5 b^3 c^2 b^{\frac{1}{3}} c^{\frac{2}{3}} = a^5 b^3 c^2 \sqrt[3]{bc^2}.$$

109 To bring $5\sqrt[4]{3}$ to an entire surd.

$$5\sqrt[4]{3} = \sqrt[4]{5^4 . 3} = \sqrt[4]{1875},$$

$$x^{\frac{2}{3}}\,y^{\frac{1}{5}}\,z^{\frac{1}{2}} = x^{\frac{20}{30}}\,y^{\frac{6}{30}}\,z^{\frac{15}{30}} = \sqrt[30]{x^{20}y^6z^{15}}.$$

110 ***To rationalise fractions having surds in their denominators.***

$$\frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7}; \quad \frac{1}{\sqrt[3]{7}} = \frac{\sqrt[3]{49}}{\sqrt[3]{7\times 49}} = \frac{\sqrt[3]{49}}{7}.$$

---

111 $$\frac{3}{9-\sqrt{80}} = \frac{3\,(9+\sqrt{80})}{81-80} = 3\,(9+\sqrt{80})$$

since $(9-\sqrt{80})\,(9+\sqrt{80}) = 81-80$, by (1).

---

112 $$\frac{1}{1+2\sqrt{3}-\sqrt{2}} = \frac{1+2\sqrt{3}+2}{(1+2\sqrt{3})^2-2} = \frac{3+2\sqrt{3}}{11+4\sqrt{3}}$$
$$= \frac{(3+2\sqrt{3})\,(11-4\sqrt{3})}{121-48} = \frac{9+10\sqrt{3}}{73}.$$

---

113 $$\frac{1}{\sqrt[3]{3}-\sqrt{2}} = \frac{1}{3^{\frac{1}{3}}-2^{\frac{1}{2}}}.$$

Put $3^{\frac{1}{3}}=x$, $2^{\frac{1}{2}}=y$, and take 6 the L. C. M. of the denominators 2 and 3, then

$$\frac{1}{x-y} = \frac{x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5}{x^6-y^6}, \text{ by (4);}$$

therefore $$\frac{1}{3^{\frac{1}{3}}-2^{\frac{1}{2}}} = \frac{3^{\frac{5}{3}}+3^{\frac{4}{3}}2^{\frac{1}{2}}+3^{\frac{3}{3}}2^{\frac{2}{2}}+3^{\frac{2}{3}}2^{\frac{3}{2}}+3^{\frac{1}{3}}2^{\frac{4}{2}}+2^{\frac{5}{2}}}{3^{\frac{6}{3}}-2^{\frac{6}{2}}}$$
$$= 3\sqrt[3]{9}+3\sqrt[6]{72}+6+2\sqrt[6]{648}+4\sqrt[3]{3}+4\sqrt{2}.$$

---

114 $\dfrac{1}{\sqrt[3]{3}+\sqrt{2}}$. Here the result will be the same as in the last example if the signs of the even terms be changed. [See 5.

---

115 A surd cannot be partly rational; that is, $\sqrt{a}$ cannot be equal to $\sqrt{b} \pm c$. Proved by squaring.

116 The product of two unlike surds is irrational;

$$\sqrt{7} \times \sqrt{3} = \sqrt{21}, \text{ an irrational quantity.}$$

117 The sum or difference of two unlike surds cannot produce a single surd; that is, $\sqrt{a}+\sqrt{b}$ cannot be equal to $\sqrt{c}$. By squaring.

118 If $a+\sqrt{m} = b+\sqrt{n}$; then $a=b$ and $m=n$.

Theorems (115) to (118) are proved indirectly.

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119 If $$\sqrt{a+\sqrt{b}} = \sqrt{x}+\sqrt{y};$$

then $$\sqrt{a-\sqrt{b}} = \sqrt{x}-\sqrt{y}.$$

By squaring and by (118).

120 To express in two terms $\sqrt{7+2\sqrt{6}}$.

Let $\sqrt{7+2\sqrt{6}} = \sqrt{x}+\sqrt{y}$

then $x+y = 7$ .... by squaring and by (118),

and $x-y = \sqrt{7^2-(2\sqrt{6})^2} = \sqrt{49-24} = 5$, by (119);

$\therefore\ x = 6$ and $y = 1$.

Result $\sqrt{6}+1$.

General formula for the same—

121 $\sqrt{a\pm\sqrt{b}} = \sqrt{\tfrac{1}{2}(a+\sqrt{a^2-b})} \pm \sqrt{\tfrac{1}{2}(a-\sqrt{a^2-b})}$.

Observe that no simplification is effected unless a^2-b is a perfect square.

---

122 To simplify $\sqrt[3]{a+\sqrt{b}}$.

Assume $\sqrt[3]{a+\sqrt{b}} = x+\sqrt{y}$

Let $c = \sqrt[3]{a^2-b}$

Then x must be found by trial from the cubic equation

$$4x^3-3cx = a$$

and $y = x^2-c$

No simplification is effected unless a^2-b is a perfect cube.

Ex. 1: $\sqrt[3]{7+5\sqrt{2}} = x+\sqrt{y}$

$c = \sqrt[3]{49-50} = -1$

$4x^3+3x = 7;\ \therefore\ x = 1,\ y = 2.$

Result $1+\sqrt{2}$.

Ex. 2: $\sqrt[3]{9\sqrt{3}-11\sqrt{2}} = \sqrt{x}+\sqrt{y}$, two different surds.

Cubing, $9\sqrt{3}-11\sqrt{2} = x\sqrt{x}+3x\sqrt{y}+3y\sqrt{x}+y\sqrt{y}$

$$\therefore\ \left.\begin{aligned} 9\sqrt{3} &= (x+3y)\sqrt{x} \\ 11\sqrt{2} &= (3x+y)\sqrt{y} \end{aligned}\right\} \qquad (118)$$

$\therefore\ x = 3$ and $y = 2$.

123 To simplify $\sqrt{(12+4\sqrt{3}+4\sqrt{5}+2\sqrt{15})}$.

Assume $\sqrt{(12+4\sqrt{3}+4\sqrt{5}+2\sqrt{5})} = \sqrt{x}+\sqrt{y}+\sqrt{z}$.

Square, and equate corresponding surds.

Result $\sqrt{3}+\sqrt{4}+\sqrt{5}$.

---

124 To express $\sqrt[n]{A \pm B}$ in the form of two surds, where A and B are one or both quadratic surds and n is odd. Take q such that $q(A^2-B^2)$ may be a perfect n^{th} power, say p^n, by (361). Take s and t the nearest integers to $\sqrt[n]{q(A+B)^2}$ and $\sqrt[n]{q(A-B)^2}$, then

$$\sqrt[n]{A \pm B} = \frac{1}{2\sqrt[2n]{q}}\{\sqrt{s+t+2p} \pm \sqrt{s+t-2p}\}.$$

EXAMPLE: To reduce $\sqrt[5]{89\sqrt{3}+109\sqrt{2}}$.

Here $A = 89\sqrt{3}$, $B = 109\sqrt{2}$,

$A^2-B^2 = 1$; $\therefore p = 1$ and $q = 1$.

$\left.\begin{array}{l}\sqrt[5]{q(A+B)^2} = 9+f \\ \sqrt[5]{q(A-B)^2} = 1-f\end{array}\right\}$, f being a proper fraction; $\therefore s = 9$, $t = 1$.

Result $\frac{1}{2}(\sqrt{9+1+2} \pm \sqrt{9+1-2}) = \sqrt{3}+\sqrt{2}$.

BINOMIAL THEOREM.

125 $(a+b)^n =$

$$a^n+na^{n-1}b+\frac{n(n-1)}{\underline{|2}}a^{n-2}b^2+\frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}b^3+\&c.$$

126 General or $(r+1)^{th}$ term,

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{|r}}a^{n-r}b^r$$

127 or $\dfrac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}a^{n-r}b^r$

if n be a positive integer.

If b be negative, the signs of the even terms will be changed.

If n be negative the expansion reduces to

128 $$(a+b)^{-n} =$$

$$a^{-n}-na^{-n-1}b+\frac{n(n+1)}{\lfloor 2}a^{-n-2}b^2-\frac{n(n+1)(n+2)}{\lfloor 3}a^{-n-3}b^3+\&c.$$

129 General term,

$$(-1)^r\frac{n(n+1)(n+2)\dots(n+r-1)}{\lfloor r}a^{-n-r}b^r. \quad \text{[See 98.}$$

Euler's proof.—Let the expansion of $(1+x)^n$, as in (125), be called $f(n)$. Then it may be proved by Induction that the equation $$f(m)\times f(n)=f(m+n) \quad \dots\dots\dots\dots\dots\dots (1)$$ is true when m and n are integers, and therefore universally true; because the *form* of an algebraical product is not altered by changing the letters involved into fractional or negative quantities. Hence

$$f(m+n+p+\&c.)=f(m)\times f(n)\times f(p), \&c.$$

Put $m=n=p=\&c.$ to k terms, each equal $\frac{h}{k}$, and the theorem is proved for a fractional index.

Again, put $-n$ for m in (1); thus, whatever n may be,

$$f(-n)\times f(n)=f(0)=1,$$

which proves the theorem for a negative index.

130 For the greatest term in the expansion of $(a+b)^n$, take

$$r = \text{the integral part of } \quad \frac{(n+1)b}{a+b} \quad \text{or} \quad \frac{(n-1)b}{a-b},$$

according as n is positive or negative.

But if b be greater than a, and n negative or fractional, the terms increase without limit.

EXAMPLES.

Required the 40th term of $\left(1-\frac{2x}{3}\right)^{42}$.

Here $r=39$; $a=1$; $b=-\frac{2x}{3}$; $n=42$.

By (127), the term will be

$$\frac{\lfloor 42}{\lfloor 3\,\lfloor 39}\left(-\frac{2x}{3}\right)^{39}=-\frac{42.41.40}{1.2.3}\left(\frac{2x}{3}\right)^{39} \text{ by (96).}$$

Required the 31st term of $(a-x)^{-4}$.

Here $r=30$; $b=-x$; $n=-4$.

By (129), the term is

$$(-1)^{30}\frac{4.5.6\ldots30.31.32.33}{1.2.3\ldots30}a^{-34}(-x)^{30}=\frac{31.32.33}{1.2.3}\cdot\frac{x^{30}}{a^{34}} \text{ by (98).}$$

---

131 Required the greatest term in the expansion of $\dfrac{1}{(1+x)^5}$ when $x=\frac{14}{17}$.

$\dfrac{1}{(1+x)^5}=(1+x)^{-5}$. Here $n=5$, $a=1$, $b=x$ in the formula

$$\frac{(n-1)\,b}{a-b}=\frac{5\times\frac{14}{17}}{1-\frac{14}{17}}=23\tfrac{1}{3}$$

therefore $r=23$, by (130), and the greatest term

$$=(-1)^5\frac{5.6.7\ldots27}{1.2.3\ldots23}\left(\frac{14}{17}\right)^{23}=-\frac{24.25.26.27}{1.2.3.4}\left(\frac{14}{17}\right)^{23}.$$

---

132 Find the first negative term in the expansion of $(2a+3b)^{\frac{17}{3}}$.

We must take r the first integer which makes $n-r+1$ negative; therefore $r>n+1=\frac{17}{3}+1=6\frac{2}{3}$; therefore $r=7$. The term will be

$$\frac{\frac{17}{3}.\frac{14}{3}.\frac{11}{3}.\frac{8}{3}.\frac{5}{3}.\frac{2}{3}\left(-\frac{1}{3}\right)}{\underline{|7}}(2a)^{-\frac{4}{3}}(3b)^7 \text{ by (126)}$$

$$=-\frac{17.14.11.8.5.2.1}{\underline{|7}}\,\frac{b^7}{(2a)^{\frac{4}{3}}}.$$

---

133 Required the coefficient of x^{34} in the expansion of $\left(\dfrac{2+3x}{2-3x}\right)^2$.

$$\frac{(2+3x)^2}{(2-3x)^2}=(2+3x)^2(2-3x)^{-2}=\left(\frac{2+3x}{2}\right)^2\left(1-\frac{3}{2}x\right)^{-2}$$

$$=\left(1+6x+\frac{9}{4}x^2\right)\left\{1+2\left(\frac{3x}{2}\right)+\frac{2.3}{1.2}\left(\frac{3x}{2}\right)^2+\ldots\ldots\right.$$

$$\left.\ldots\ldots+33\left(\frac{3x}{2}\right)^{32}+34\left(\frac{3x}{2}\right)^{33}+35\left(\frac{3x}{2}\right)^{34}+\&\text{c.}\ldots\ldots\right\}$$

The three terms last written being those which produce x^{34} after multiplying by the factor $(1+6x+\frac{9}{4}x^2)$; for we have

$$\frac{9}{4}x^2\times33\left(\frac{3x}{2}\right)^{32}+6x\times34\left(\frac{3x}{2}\right)^{33}+1\times35\left(\frac{3x}{2}\right)^{34}$$

giving for the coefficient of x^4 in the result

$$\frac{297}{4}\left(\frac{3}{2}\right)^{32}+204\left(\frac{3}{2}\right)^{33}+35\left(\frac{3}{2}\right)^{34}=459\left(\frac{3}{2}\right)^{32}$$

The coefficient of x^n will in like manner be $459\left(\frac{3}{2}\right)^{n-2}$.

134 To write the coefficient of x^{3m+1} in the expansion of $\left(x^2-\frac{1}{x^2}\right)^{2n+1}$.

The general term is

$$\frac{|\underline{2n+1}}{|\underline{2n+1-r}\,|\underline{r}}\,x^{2(2n-r+1)}\cdot\frac{1}{x^{2r}}=\frac{|\underline{2n+1}}{|\underline{2n+1-r}\,|\underline{r}}\,x^{4n-4r+2}$$

Equate $4n-4r+2$ to $3m+1$, thus

$$r=\frac{4n-3m+1}{4}$$

Substitute this value of r in the general term; the required coefficient becomes

$$\frac{|\underline{2n+1}}{|\underline{\frac{4n+3m+3}{4}}\;|\underline{\frac{4n-3m+1}{4}}}$$

The value of r shows that there is no term in x^{3m+1} unless $\frac{4n-3m+1}{4}$ is an integer.

---

135 An approximate value of $(1+x)^n$ when x is small is $1+nx$, by (125), neglecting x^2 and higher powers of x.

---

136 Ex.—An approximation to $\sqrt[3]{999}$ by Bin. Th. (125) is obtained from the first two or three terms of the expansion of

$$(1000-1)^{\frac{1}{3}}=10-\tfrac{1}{3}\,.\,1000^{-\frac{2}{3}}=10-\tfrac{1}{300}=9\tfrac{299}{300}\text{ nearly.}$$

MULTINOMIAL THEOREM.

The general term in the expansion of $(a+bx+cx^2+\&c.)^n$ is

137 $$\frac{n(n-1)(n-2)\dots(p+1)}{|\underline{q}\;|\underline{r}\;|\underline{s}\dots}\,a^pb^qc^r\dots x^{q+2r+3s+\dots}$$

where $$p+q+r+s+\&c.=n,$$

and the number of terms p, q, r, &c. corresponds to the number of terms in the given multinomial.

p is integral, fractional, or negative, according as n is one or the other.

If n be an integer, (137) may be written

138 $$\frac{|\underline{n}}{|\underline{p}\;|\underline{q}\;|\underline{r}\dots}\,a^pb^qc^r\dots x^{q+2r+3s}.$$

[Deduced from the Bin. Theor.

Ex. 1.—To write the coefficient of a^3bc^5 in the expansion of $(a+b+c+d)^{10}$. Here put $n=10$, $x=1$, $p=3$, $q=1$, $r=5$, $s=0$ in (138).

Result $$\frac{\underline{|10}}{\underline{|3}\,\underline{|5}} = \frac{10 \,.\, 9 \,.\, 8}{1 \,.\, 2 \,.\, 3}.$$

Ex. 2.—To obtain the coefficient of x^8 in the expansion of

$$(1-2x+3x^2-4x^3)^4.$$

Here, comparing with (137), we have $a=1$, $b=-2$, $c=3$, $d=-4$,

$$p+q+r+s=4,$$
$$q+2r+3s=8,$$

1	0	1	2
0	2	0	2
0	1	2	1
0	0	4	0

The numbers 1, 0, 1, 2 are particular values of p, q, r, s respectively, which satisfy the two equations given above.

0, 2, 0, 2 are another set of values which also satisfy those equations; and the four rows of numbers constitute all the solutions. In forming these rows always try the highest possible numbers on the right first.

Now substitute each set of values of p, q, r, s in formula (138) successively as under:

$$\frac{\underline{|4}}{\underline{|2}}\, 1^1\,(-2)^0\, 3^1\,(-4)^2 = 576$$

$$\frac{\underline{|4}}{\underline{|2}\,\underline{|2}}\, 1^0\,(-2)^2\, 3^0\,(-4)^2 = 384$$

$$\frac{\underline{|4}}{\underline{|2}}\, 1^0\,(-2)^1\, 3^2\,(-4)^1 = 864$$

$$\frac{\underline{|4}}{\underline{|4}}\, 1^0\,(-2)^0\, 3^4\,(-4)^0 = 81$$

Result 1905

Ex. 3.—Required the coefficient of x^4 in $(1+2x-4x^2-2x^3)^{-\frac{1}{2}}$.

Here $a=1$, $b=2$, $c=-4$, $d=-2$, $n=-\frac{1}{2}$; and the two equations are

$$p+q+r+s=-\tfrac{1}{2},$$
$$q+2r+3s=4,$$

$-\frac{5}{2}$	1	0	1
$-\frac{5}{2}$	0	2	0
$-\frac{7}{2}$	2	1	0
$-\frac{9}{2}$	4	0	0

Employing formula (137), the remainder of the work stands as follows:

$$\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)1^{-\frac{5}{2}}\,2^1(-4)^0(-2)^1 = -\ 3$$

$$\frac{1}{\lfloor 2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)1^{-\frac{5}{2}}\,2^0(-4)^2(-2)^0 = \quad 6$$

$$\frac{1}{\lfloor 2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)1^{-\frac{7}{2}}\,2^2(-4)^1(-2)^0 = \quad 15$$

$$\frac{1}{\lfloor 4}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)1^{-\frac{9}{2}}\,2^4(-4)^0(-2)^0 = \quad \frac{35}{8}$$

Result $22\frac{3}{8}$

---

139 The number of terms in the expansion of the multinomial $(a+b+c+\text{to } n \text{ terms})^r$ is the same as the number of homogeneous products of n things of r dimensions. See (97 and 98).

The greatest coefficient in the expansion of $(a+b+c+\text{to } m \text{ terms})^n$, n being an integer, is

140 $$\frac{\lfloor n}{(\lfloor q)^m (q+1)^k} \text{ where } qm+k=n.$$

Obtained by making the denominator in (138) as small as possible.

LOGARITHMS.

---

142 $\log_a N = x$ signifies that $a^x = N$, or

Def.—*The logarithm of a number is the power to which the base must be raised to produce that number.*

---

143 $$\log_a a = 1, \quad \log 1 = 0.$$

144 $$\log MN = \log M + \log N.$$

$$\log \frac{M}{N} = \log M - \log N.$$

$$\log (M)^n = n \log M.$$

$$\log \sqrt[n]{M} = \frac{1}{n}\log M.$$

[142

145 $$\log_b a = \frac{\log_c a}{\log_c b}.$$

That is—*The logarithm of a number to any base is equal to the logarithm of the number divided by the logarithm of the base*, the two last named logarithms being taken to any the same base at pleasure.

146 $$\log_b a = \frac{1}{\log_a b}.\quad \text{Put } c=a \text{ in (145).}$$

147 $$\log_{10} N = \frac{\log_e N}{\log_e 10} \text{ by (145).}$$

148 $$\frac{1}{\log_e 10} = \cdot 43429448 \ldots$$

is called the modulus of the common system of logarithms; that is, the factor which will convert logarithms of numbers calculated to the base e into the corresponding logarithms to the base 10. See (154).

EXPONENTIAL THEOREM.

149 $$a^x = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \&c.,$$

where $$c = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

PROOF: $a^x = \{1+(a-1)\}^x$. Expand this by Binomial Theorem, and collect the coefficients of x; thus c is obtained. Assume c_2, c_3, &c. as the coefficients of the succeeding powers of x, and with this assumption write out the expansions of a^x, a^y, and a^{x+y}. Form the product of the first two series, which product must be equivalent to the third. Therefore equate the coefficient of x in this product with that in the expansion of a^{x+y}. In the identity so obtained equate the coefficients of the successive powers of y to determine c_2, c_3, &c.

Let e be that value of a which makes $c=1$, then

150 $$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \&c.$$

Hence, by putting $x=1$,

151 $$e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \&c.$$

$$= 2{\cdot}718281828 \ldots\ldots \text{ an incommensurable quantity.}$$

See (295).

152 By making $x=1$ in (149), and $x=c$ in (150), we obtain $a = e^c$; that is, $c = \log_e a$. Therefore by (149)

154 $$\log_e a = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

155 $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$$

156 $$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \&c.$$ [154

157 $$\therefore \log\frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \&c.\right\}$$

Put $\dfrac{m-1}{m+1}$ for x in (157); thus,

158 $$\log m = 2\left\{\frac{m-1}{m+1} + \frac{1}{3}\left(\frac{m-1}{m+1}\right)^3 + \frac{1}{5}\left(\frac{m-1}{m+1}\right)^5 + \&c.\right\}$$

Put $\dfrac{1}{2n+1}$ for x in (157); thus,

159 $\log(n+1) - \log n$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \&c.\right\}$$

CONTINUED FRACTIONS AND CONVERGENTS.

160 To find convergents to $3{\cdot}14159 = \frac{314159}{100000}$. Proceed as in the rule for H. C. F.

7	100000	314159	3
	99113	300000	
1	887	14159	15
	854	887	
1	33	5289	
	29	4435	
4	4	854	25
	4	66	
		194	
		165	
		29	7
		28	
		1	

The continued fraction is

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \&c.}};$$

or, as it is more conveniently written,

$$3 + \frac{1}{7+}\,\frac{1}{15+}\;\&c.$$

The convergents are formed as follows:—

3	7	15	1	25	1	7	4
$\frac{3}{1}$,	$\frac{22}{7}$,	$\frac{333}{106}$,	$\frac{355}{113}$,	$\frac{9208}{2931}$,	$\frac{9563}{3044}$,	$\frac{76149}{24239}$,	$\frac{314159}{100000}$.

161 Rule.—Write the quotients in a row, and the first two convergents at sight (in the example 3 and $3+\frac{1}{7}$). Multiply the numerator of any convergent by the next quotient, and add the previous numerator. The result is the numerator of the next convergent. Proceed in the same way to determine the denominator. The last convergent should be the original fraction in its lowest terms.

162 *Formula for forming the convergents.*

If $\frac{p_{n-2}}{q_{n-2}}$, $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ are any consecutive convergents, and a_{n-2}, a_{n-1}, a_n the corresponding quotients; then

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

The n^{th} convergent is therefore

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} \equiv F_n.$$

The true value of the continued fraction will be expressed by

163 $$F = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}}$$

in which a'_n is the complete quotient or value of the continued fraction commencing with a_n.

164 $p_n q_{n-1} - p_{n-1} q_n = \pm 1$ alternately, by (162).

The convergents are alternately greater and less than the original fraction, and are always in their lowest terms.

165 The difference between F_n and the true value of the continued fraction is

$$< \frac{1}{q_n q_{n+1}} \quad \text{and} \quad > \frac{1}{q_n (q_n + q_{n+1})}$$

and this difference therefore diminishes as n increases. Proved by taking the difference, $\frac{p_n}{q_n} - \frac{a' p_{n+1} + p_n}{a' q_{n+1} + q_n}$ (163).

Also F is nearer the true value than any other fraction with a less denominator.

166 $F_n F_{n+1}$ is greater or less than F^2 according as F_n is greater or less than F_{n+1}.

General Theory of Continued Fractions.

167

First class of continued fraction.	Second class of continued fraction.
$F = \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 + \&c.}$	$V = \frac{b_1}{a_1 -} \frac{b_2}{a_2 -} \frac{b_3}{a_3 - \&c.}$

a_1, b_1, &c. are taken as positive quantities.

$\frac{b_1}{a_1}$, $\frac{b_2}{a_2}$, &c. are termed *components* of the continued fraction. If the components be infinite in number, the continued fraction is said to be infinite.

Let the successive convergents be denoted by

$$\frac{p_1}{q_1} = \frac{b_1}{a_1}; \quad \frac{p_2}{q_2} = \frac{b_1}{a_1+}\,\frac{b_2}{a_2}; \quad \frac{p_3}{q_3} = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3}; \text{ and so on.}$$

168 The law of formation of the convergents is,

For F,	For V,
$\begin{cases} p_n = a_n p_{n-1} + b_n p_{n-2} \\ q_n = a_n q_{n-1} + b_n q_{n-2} \end{cases}$	$\begin{cases} p_n = a_n p_{n-1} - b_n p_{n-2} \\ q_n = a_n q_{n-1} - b_n q_{n-2} \end{cases}$

Proved by Induction.

The relation between the successive differences of the convergents is, by (168),

169 $$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \mp \frac{b_{n+1} q_{n-1}}{q_{n+1}} \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right)$$

Take the $-$ sign for F, and the $+$ for V.

170 $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} b_1 b_2 b_3 \dots b_n$ (by 168).

171 The odd convergents for F, $\frac{p_1}{q_1}$, $\frac{p_3}{q_3}$, &c., continually decrease, and the even convergents, $\frac{p_2}{q_2}$, $\frac{p_4}{q_4}$, &c., continually increase. (167.)

Every odd convergent is greater, and every even convergent is less, than all following convergents. (169.)

172 Def.—If the difference between consecutive convergents diminishes without limit, the infinite continued fraction is said to be *definite*. If the same difference tends to a fixed value greater than zero, the infinite continued fraction is *indefinite;* the odd convergents tending to one value, and the even convergents to another.

173 F is definite if the ratio of every quotient to the next component is greater than a fixed quantity.

Apply (169) successively.

174 F is incommensurable when the components are all proper fractions and infinite in number.

Indirectly, and by (168).

175 If a be never less than $b+1$, the convergents of V are all positive proper fractions, increasing in magnitude, p_n and q_n also increasing with n. By (167) and (168).

176 If, in this case, V be infinite, it is also definite, being $=1$, if a always $=b+1$ while b is less than 1, (175); and being less than 1, if a is ever greater than $b+1$. By (180).

177 V is incommensurable when it is less than 1, and the components are all proper fractions and infinite in number.

180 If in the continued fraction V (167), we have $a_n = b_n + 1$ always; then, by (168),

$$p_n = b_1 + b_1 b_2 + b_1 b_2 b_3 + \dots \text{ to } n \text{ terms, and } q_n = p_n + 1.$$

---

181 If, in the continued fraction F, a_n and b_n are constant and equal, say, to a and b respectively; then p_n and q_n are respectively equal to the coefficients of x^{n-1} in the expansions

of $$\frac{b}{1-ax-bx^2} \quad \text{and} \quad \frac{a+bx}{1-ax-bx^2}.$$

For p_n and q_n are the n^{th} terms of two recurring series.

See (168) and (251).

---

182 ***To convert a Series into a Continued Fraction.***

The series $$\frac{1}{u} + \frac{x}{u_1} + \frac{x^2}{u_2} + \dots + \frac{x^n}{u_n}$$

is equal to a continued fraction V (167), with $n+1$ components; the first, second, and $n+1^{\text{th}}$ components being

$$\frac{1}{u}, \quad \frac{u^2 x}{u_1 + ux}, \quad \dots \quad \frac{u_{n-1}^2 x}{u_n + u_{n-1} x}$$

Proved by Induction.

183 The series

$$\frac{1}{v}+\frac{x}{vv_1}+\frac{x^2}{vv_1v_2}+\ldots+\frac{x^n}{vv_1v_2\ldots v_n}$$

is equal to a continued fraction V (167), with $n+1$ components, the first, second, and $n+1^{\text{th}}$ components being

$$\frac{1}{v},\ \frac{vx}{v_1+x},\ \ldots\ldots\ \frac{v_{n-1}x}{v_n+x}.$$ [Proved by Induction.

184 The sign of x may be changed in either of the statements in (182) or (183).

185 Also, if any of these series are convergent and infinite, the continued fractions become infinite.

186 ***To find the value of a continued fraction with recurring quotients.***

Let the continued fraction be

$$x=\frac{b_1}{a_1+}\ldots+\frac{b_n}{a_n+y}\quad\text{where}\quad y=\frac{b_{n+1}}{a_{n+1}+}\ldots+\frac{b_{n+m}}{a_{n+m}+y}$$

so that there are m recurring quotients. Form the n^{th} convergent for x, and the m^{th} for y. Then, by substituting the complete quotients a_n+y for a_n, and $a_{n+m}+y$ for a_{n+m} in (168), two equations are obtained of the forms

$$x=\frac{Ay+B}{Cy+D}\quad\text{and}\quad y=\frac{Ey+F}{Gy+H}$$

from which, by eliminating y, a quadratic equation for determining x is obtained.

187 If

$$\frac{b_1}{a_1+}\ldots+\frac{b_n}{a_n+}$$

be a continued fraction, and

$$\frac{p_1}{q_1},\ \ldots\ldots\ \frac{p_n}{q_n}$$

the corresponding first n convergents; then $\frac{q_{n-1}}{q_n}$, developed by (168), produces the continued fraction

$$\frac{1}{a_n +} \frac{b_n}{a_{n-1} +} \frac{b_{n-1}}{a_{n-2} +} \dots + \frac{b_3}{a_2 +} \frac{b_2}{a_1}$$

the quotients being the same but in reversed order.

INDETERMINATE EQUATIONS.

188 Given $$ax + by = c$$

free from fractions, and α, β integral values of x and y which satisfy the equation, the complete integral solution is given by

$$x = \alpha - bt$$
$$y = \beta + at$$

where t is any integer.

EXAMPLE.—Given $5x + 3y = 112$.

Then $x = 20$, $y = 4$ are values;

$$\therefore \quad \left.\begin{aligned} x &= 20 - 3t \\ y &= 4 + 5t \end{aligned}\right\}$$

The values of x and y may be exhibited as under:

$t =$	-2	-1	0	1	2	3	4	5	6	7
$x =$	26	23	20	17	14	11	8	5	2	-1
$y =$	-6	-1	4	9	14	19	24	29	34	39

For solutions in positive integers t must lie between $\frac{20}{3} = 6\frac{2}{3}$ and $-\frac{4}{5}$; that is, t must be 0, 1, 2, 3, 4, 5, or 6, giving 7 positive integral solutions.

189 If the equation be

$$ax - by = c$$

the solutions are given by

$$x = \alpha + bt$$
$$y = \beta + at$$

EXAMPLE: $4x-3y = 19.$

Here $x = 10$, $y = 7$ satisfy the equation;

$$\therefore \quad \left.\begin{aligned} x &= 10+3t \\ y &= \ \ 7+4t \end{aligned}\right\} \text{ furnish all the solutions.}$$

The simultaneous values of t, x, and y will be as follows:—

$t = -5$	-4	-3	-2	-1	0	1	2	3
$x = -5$	-2	1	4	7	10	13	16	19
$y = -13$	-9	-5	-1	3	7	11	15	19

The number of positive integral solutions is infinite, and the least positive integral values of x and y are given by the limiting value of t, viz.,

$$t > -\frac{10}{3} \text{ and } t > -\frac{7}{4};$$

that is, t must be $-1, 0, 1, 2, 3$, or greater.

190 If two values, α and β, cannot readily be found by inspection, as, for example, in the equation

$$17x+13y = 14900,$$

divide by the least coefficient, and equate the remaining fractions to t, an integer; thus

$$y + x + \frac{4x}{13} = 1146 + \frac{2}{13} \text{..........................(1)};$$

$$\therefore \quad 4x-2 = 13t.$$

Repeat the process; thus

$$x - \frac{2}{4} = 3t + \frac{t}{4},$$

$$\therefore \quad t+2 = 4u.$$

Put $u = 1,$

$$\therefore \quad t = 2,$$

$$x = \frac{13t+2}{4} = 7 = \alpha;$$

and $y+x+t = 1146$, by (1),

$$\therefore \quad y = 1146-7-2 = 1137 = \beta.$$

The general solution will be

$$x = \quad 7-13t,$$
$$y = 1137+17t.$$

Or, changing the sign of t for convenience,

$$x = \quad 7+13t,$$
$$y = 1137-17t.$$

Here the number of solutions in positive integers is equal to the number of integers lying between $-\frac{7}{13}$ and $\frac{1137}{17}$;

or $-\frac{7}{13}$ and $66\frac{15}{17}$; that is, 67.

---

191 Otherwise.—Two values of x and y may be found in the following manner:—

Find the nearest converging fraction to $\frac{17}{13}$ by (160).

This is $\frac{4}{3}$. By (163) we have

$$17\times 3-13\times 4=-1.$$

Multiply by 14900, and change the signs;

$$\therefore \quad 17(-44700)+13(59600)=14900;$$

which shews that we may take $\begin{cases}\alpha=-44700\\ \beta=\quad 59600\end{cases}$

and the general solution may be written

$$x=-44700+13t,$$
$$y=\quad 59600-17t.$$

This method has the disadvantage of producing high values of α and β.

---

192 The values of x and y, in positive integers, which satisfy the equation $ax\pm by=c$, form two Arithmetic Progressions, of which b and a are respectively the common differences. See examples (188) and (189).

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193 Abbreviation of the method in (169).

EXAMPLE: $11x-18y=63.$

Put $x=9z$, and divide by 9; then proceed as before.

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194 *To obtain integral solutions of* $ax+by+cz=d$.

Write the equation thus

$$ax+by=d-cz.$$

Put successive integers for z, and solve for x, y in each case.

TO REDUCE A QUADRATIC SURD TO A CONTINUED FRACTION.

195 EXAMPLE:

$$\sqrt{29} = 5 + \sqrt{29}-5 = 5 + \frac{4}{\sqrt{29}+5},$$

$$\frac{\sqrt{29}+5}{4} = 2 + \frac{\sqrt{29}-3}{4} = 2 + \frac{5}{\sqrt{29}+3},$$

$$\frac{\sqrt{29}+3}{5} = 1 + \frac{\sqrt{29}-2}{5} = 1 + \frac{5}{\sqrt{29}+2},$$

$$\frac{\sqrt{29}+2}{5} = 1 + \frac{\sqrt{29}-3}{5} = 1 + \frac{4}{\sqrt{29}+3},$$

$$\frac{\sqrt{29}+3}{4} = 2 + \frac{\sqrt{29}-5}{4} = 2 + \frac{1}{\sqrt{29}+5},$$

$$\sqrt{29}+5 = 10 + \sqrt{29}-5 = 10 + \frac{4}{\sqrt{29}+5}.$$

The quotients 5, 2, 1, 1, 2, 10 are the greatest integers contained in the quantities in the first column. The quotients now recur, and the surd $\sqrt{29}$ is equivalent to the continued fraction

$$5 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{10+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\&c.$$

The convergents to $\sqrt{29}$, formed as in (160), will be

$$\frac{5}{1},\ \frac{11}{2},\ \frac{16}{3},\ \frac{27}{5},\ \frac{70}{13},\ \frac{727}{135},\ \frac{1524}{283},\ \frac{2251}{418},\ \frac{3775}{701},\ \frac{9801}{1820}.$$

196 Note that the last quotient 10 is the greatest and twice the first, that the *second* is the first of the recurring ones, and that the recurring quotients, excluding the last, consist of pairs of equal terms, quotients equi-distant from the first and last being equal. These properties are universal. (See 204—210).

To form high convergents rapidly.

197 Suppose m the number of recurring quotients, or any multiple of that number, and let the m^{th} convergent to $\sqrt{Q}$ be represented by F_m; then the $2m^{\text{th}}$ convergent is given by the formula

$$F_{2m} = \frac{1}{2}\left\{F_m + \frac{Q}{F_m}\right\} \text{ by (203) and (210).}$$

198 For example, in approximating to $\sqrt{29}$ above, there are five recurring quotients. Take $m = 2 \times 5 = 10$; therefore, by

$$F_{20} = \frac{1}{2}\left\{F_{10} + \frac{29}{F_{10}}\right\}$$

$$F_{10} = \frac{9801}{1820}, \text{ the } 10^{\text{th}} \text{ convergent.}$$

Therefore $$F_{20} = \frac{1}{2}\left\{\frac{9801}{1820} + 29 \times \frac{1820}{9801}\right\} = \frac{192119201}{35675640},$$

the 20^{th} convergent to $\sqrt{29}$; and the labour of calculating the intervening convergents is saved.

GENERAL THEORY.

199 The process of (174) may be exhibited as follows:—

$$\frac{\sqrt{Q}+c_1}{r_1} = a_1 + \frac{r_2}{\sqrt{Q}+c_2}$$

$$\frac{\sqrt{Q}+c_2}{r_2} = a_2 + \frac{r_3}{\sqrt{Q}+c_3}$$

... ...

... ...

$$\frac{\sqrt{Q}+c_n}{r_n} = a_n + \frac{r_{n+1}}{\sqrt{Q}+c_{n+1}}$$

200 Then $$\sqrt{Q} = a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\,\&c.$$

The quotients a_1, a_2, a_3, &c. are the integral parts of the fractions on the left.

201 The equations connecting the remaining quantities are

$$c_1 = 0 \qquad r_1 = 1$$

$$c_2 = a_1 r_1 - c_1 \qquad r_2 = \frac{Q - c_2^2}{r_1}$$

$$c_3 = a_2 r_2 - c_2 \qquad r_3 = \frac{Q - c_3^2}{r_2}$$

$$\cdots \quad \cdots \quad \cdots \qquad \cdots \quad \cdots \quad \cdots$$

$$c_n = a_{n-1} r_{n-1} - c_{n-1} \qquad r_n = \frac{Q - c_n^2}{r_{n-1}}$$

The n^{th} convergent to $\sqrt{Q}$ will be

202 $$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$$ [By Induction.

The true value of $\sqrt{Q}$ is what this becomes when we substitute for a_n the complete quotient $\frac{\sqrt{Q}+c_n}{r_n}$, of which a_n is only the integral part. This gives

203 $$\sqrt{Q} = \frac{(\sqrt{Q}+c_n)\, p_{n-1} + r_n p_{n-2}}{(\sqrt{Q}+c_n)\, q_{n-1} + r_n q_{n-2}}$$

By the relations (199) to (203) the following theorems are demonstrated :—

204 All the quantities a, r, and c are positive integers.

205 The greatest c is c_2, and $c_2 = a_1$.

206 No a or r can be greater than $2a_1$.

207 If $r_n = 1$, then $c_n = a_1$.

208 For all values of n greater than 1, $a - c_n$ is $< r_n$.

209 The number of quotients cannot be greater than $2a_1^2$. The last quotient is $2a_1$, and after that the terms repeat. The first complete quotient that is repeated is $\frac{\sqrt{Q}+c_2}{r_2}$, and a_2, r_2, c_2 commence each cycle of repeated terms.

210 Let a_m, r_m, c_m be the last terms of the first cycle; then $a_{m-1}, r_{m-1}, c_{m-1}$ are respectively equal to a_2, r_2, c_2; $a_{m-2}, r_{m-2}, c_{m-2}$ are equal to a_3, r_3, c_3, and so on. By (187).

EQUATIONS.

Special Cases in the Solution of Simultaneous Equations.

211 First, with two unknown quantities.

$$\left.\begin{array}{l} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{array}\right\}. \quad x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1}, \quad y=\frac{c_1a_2-c_2a_1}{b_1a_2-b_2a_1}.$$

If the denominators vanish, we have

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}, \text{ and } x=\infty, \; y=\infty;$$

unless at the same time the numerators vanish, for then

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}; \quad x=\frac{0}{0}; \quad y=\frac{0}{0};$$

and the equations are *not independent*, one being produced by multiplying the other by some constant.

212 Next, with three unknown quantities. See (60) for the equations.

If d_1, d_2, d_3 all vanish, divide each equation by z, and we have three equations for finding the two ratios $\frac{x}{z}$ and $\frac{y}{z}$, two only of which equations are necessary, any one being deducible from the other two if the three be consistent.

213 ***To solve simultaneous equations by Indeterminate Multipliers.***

Ex.—Take the equations

$$\begin{aligned} x+2y+\;\,3z+4w&=27, \\ 3x+5y+\;\,7z+\;\,w&=48, \\ 5x+8y+10z-2w&=65, \\ 7x+6y+\;\,5z+4w&=53. \end{aligned}$$

Multiply the first by A, the second by B, the third by C, leaving one equation unmultiplied; and then add the results.

Thus $$(A+3B+5C+7)x+(2A+5B+8C+6)y$$
$$+(3A+7B+10C+5)z+(4A+B-2C+4)w$$
$$=27A+48B+65C+53.$$

To determine either of the unknowns, for instance x, equate the coefficients of the other three separately to zero, and from the three equations find A, B, C. Then

$$x=\frac{27A+48B+65C+53}{A+3B+5C+7}$$

MISCELLANEOUS EQUATIONS AND SOLUTIONS.

214 $$x^6 \pm 1 = 0.$$

Divide by x^3, and throw into factors, by (2) or (3). See also (480).

215 $$x^3-7x-6 = 0.$$

$x=-1$ is a root, by inspection; therefore $x+1$ is a factor. Divide by $x+1$, and solve the resulting quadratic.

216 $$x^3+16x = 455.$$

$$x^4+16x^2 = 455x = 65\times 7x$$

$$x^4+65x^2+\left(\frac{65}{2}\right)^2 = 49x^2+65\times 7x+\left(\frac{65}{2}\right)^2$$

$$x^2+\frac{65}{2} = 7x+\frac{65}{2}$$

$$x^2 = 7x; \quad \therefore \quad x=7.$$

RULE.—Divide the absolute term (here 455) into two factors, if possible, such that one of them, minus the square of the other, equals the coefficient of x. See (483) for general solution of a cubic equation.

217 $x^4-y^4=14560, \quad x-y=8.$

Put $x=z+v$ and $y=z-v$.

Eliminate v, and obtain a cubic in z, which solve as in (216).

218 $x^5-y^5=3093, \quad x-y=3.$

Divide the first equation by the second, and subtract from the result the fourth power of $x-y$. Eliminate (x^2+y^2), and obtain a quadratic in xy.

219 *On forming Symmetrical Expressions.*

Take, for example, the equation

$$(y-c)(z-b)=a^2.$$

To form the remaining equations symmetrical with this, write the corresponding letters in *vertical* columns, observing the circular order in which a is followed by b, b by c, and c by a. So with x, y, and z. Thus the equations become

$$(y-c)(z-b)=a^2$$
$$(z-a)(x-c)=b^2$$
$$(x-b)(y-a)=c^2$$

To solve these equations substitute

$$x=b+c+x', \quad y=c+a+y', \quad z=a+b+z';$$

and multiplying out, and eliminating y and z, we obtain

$$x=\frac{bc(b+c)-a(b^2+c^2)}{bc-ca-ab}$$

and therefore, by symmetry, the values of y and z, by the rule just given.

220
$$y^2+z^2+yz=a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$z^2+x^2+zx=b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$x^2+y^2+xy=c^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$
$$\therefore \quad 3(yz+zx+xy)^2=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4 \ldots\ldots (4).$$

Now add (1), (2), and (3), and we obtain

$$2(x+y+z)^2-3(yz+zx+xy) = a^2+b^2+c^2 \dots\dots\dots (5).$$

From (4) and (5) $(x+y+z)$ is obtained, and then (1), (2), and (3) are readily solved.

221

$$x^2-yz = a^2 \dots\dots\dots (1),$$

$$y^2-zx = b^2 \dots\dots\dots (2),$$

$$z^2-xy = c^2 \dots\dots\dots (3).$$

Multiply (2) by (3), and subtract (1)^2.

Result $\quad x(3xyz-x^3-y^3-z^3) = b^2c^2-a^4$

$$\therefore \quad \frac{x}{b^2c^2-a^4} = \frac{y}{c^2a^2-b^4} = \frac{z}{a^2b^2-c^4} = \lambda \dots\dots\dots (4).$$

Obtain λ^2 by proportion as a fraction with numerator

$$= x^2-yz = a^2$$

222

$$x = cy+bz \dots\dots\dots (1),$$

$$y = az+cx \dots\dots\dots (2),$$

$$z = bx+ay \dots\dots\dots (3).$$

Eliminate a between (2) and (3), and substitute the value of x from equation (1).

Result $\quad \dfrac{y^2}{1-b^2} = \dfrac{z^2}{1-c^2} = \dfrac{x^2}{1-a^2}$

IMAGINARY EXPRESSIONS.

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223 The following are conventions :—

That $\sqrt{(-a^2)}$ is equivalent to $a\sqrt{(-1)}$; that $a\sqrt{(-1)}$ vanishes when a vanishes; that the symbol $a\sqrt{(-1)}$ is subject to the ordinary rules of Algebra. $\sqrt{(-1)}$ is denoted by i.

224 If $\alpha+i\beta=\gamma+i\delta$; then $\alpha=\gamma$ and $\beta=\delta$.

225 $\alpha+i\beta$ and $\alpha-i\beta$ are conjugate expressions; their product $=\alpha^2+\beta^2$.

226 The sum and product of two conjugate expressions are both real, but their difference is imaginary.

---

227 The modulus is $+\sqrt{\alpha^2+\beta^2}$.

228 If the modulus vanishes, α and β must vanish.

229 If two imaginary expressions are equal, their moduli are equal, by (224).

230 The modulus of the product of two imaginary expressions is equal to the product of their moduli.

231 Also the modulus of the quotient is equal to the quotient of their moduli.

METHOD OF INDETERMINATE COEFFICIENTS.

232 If $A+Bx+Cx^2+\ldots=A'+B'x+C'x^2+\ldots$ be an equation which holds for all values of x, the coefficients A, B, &c., not involving x; then $A=A'$, $B=B'$, $C=C'$, &c.; that is, the coefficients of like powers of x must be equal. Proved by putting $x=0$, and dividing by x alternately. See (234) for an example.

---

233 METHOD OF PROOF BY INDUCTION.

Ex.—To prove that

$$1+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

Assume $$1+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6};$$

$$\therefore \quad 1+2^2+3^2+\ldots+n^2+(n+1)^2 = \frac{n(n+1)(2n+1)}{6}+(n+1)^2$$

$$= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} = \frac{(n+1)\{n(2n+1)+6(n+1)\}}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} = \frac{n'(n'+1)(2n'+1)}{6}$$

where n' is written for $n+1$;

$$\therefore \quad 1+2^2+3^2+\ldots+n'^2 = \frac{n'(n'+1)(2n'+1)}{6}.$$

It is thus proved that *if the formula be true for n it is also true for n+1.*

But the formula *is* true when $n=2$ or 3, as may be shewn by actual trial; therefore it is true when $n=4$; therefore also when $n=5$, and so on; therefore universally true.

---

234 Ex.—The same theorem proved by the method of Indeterminate coefficients.

Assume

$$1+2^2+3^2+\ldots+n^2 = A+Bn+Cn^2+Dn^3+\&c.;$$

$$\therefore 1+2^2+3^2+\ldots+n^2+(n+1)^2 = A+B(n+1)+C(n+1)^2+D(n+1)^3+\&c.;$$

therefore, by subtraction,

$$n^2+2n+1 = B+C(2n+1)+D(3n^2+3n+1),$$

writing no terms in this equation which contain higher powers of n than the highest which occurs on the left-hand side, for the coefficients of such terms may be shewn to be separately equal to zero.

Now equate the coefficients of like powers of n; thus

$$3D=1, \quad \therefore \quad D=\frac{1}{3};$$

$$2C+3D=2, \quad \therefore \quad C=\frac{1}{2}, \quad \text{and} \quad A=0;$$

$$B+C+D=1, \quad \therefore \quad B=\frac{1}{6};$$

therefore the sum of the series is equal to

$$\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3} = \frac{n(n+1)(2n+1)}{6}.$$

PARTIAL FRACTIONS.

In the resolution of a fraction into partial fractions four cases present themselves, which are illustrated in the following examples,

235 First.—When there are no repeated factors in the denominator of the given fraction.

Ex.—To resolve $\frac{3x-2}{(x-1)(x-2)(x-3)}$ into partial fractions.

Assume $$\frac{3x-2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3};$$

$$\therefore \quad 3x-2 = A(x-2)(x-3)+B(x-3)(x-1)+C(x-1)(x-2).$$

Since A, B, and C do not contain x, and this equation is true for all values of x, put $x = 1$; then

$$3-2 = A(1-2)(1-3), \text{ from which } A = \frac{1}{2}.$$

Similarly, if x be put $= 2$, we have

$$6-2 = B(2-3)(2-1); \quad \therefore \quad B = -4;$$

and, putting $x = 3$,

$$9-2 = C(3-1)(3-2); \quad \therefore \quad C = \frac{7}{2}.$$

Hence $$\frac{3x-2}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{4}{x-2} + \frac{7}{2(x-3)}.$$

236 Secondly.—When there is a repeated factor.

Ex.—Resolve into partial fractions $\frac{7x^3-10x^2+6x}{(x-1)^3(x+2)}$.

Assume $$\frac{7x^3-10x^2+6x}{(x-1)^3(x+2)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)} + \frac{D}{x+2}.$$

These forms are necessary and sufficient. Multiplying up, we have

$$7x^3-10x^2+6x = A(x+2)+B(x-1)(x+2)+C(x-1)^2(x+2)+D(x-1)^3 \quad \text{............ (1)}.$$

Make $x=1$; $\therefore$ $7-10+6 = A(1+2)$; $\therefore$ $A = 1$.

Substitute this value of A in (1); thus

$$7x^3-10x^2+5x-2 = B(x-1)(x+2)+C(x-1)^2(x+2)+D(x-1)^3.$$

Divide by $x-1$; thus

$$7x^2-3x+2 = B(x+2)+C(x-1)(x+2)+D(x-1)^2 \quad \text{......... (2)}.$$

Make $x=1$ again, $7-3+2 = B(1+2)$; $\therefore$ $B = 2$.

Substitute this value of B in (2), and we have

$$7x^2-5x-2 = C(x-1)(x+2)+D(x-1)^2.$$

Divide by $x-1$, $\quad 7x+2 = C(x+2)+D(x-1)$ .... (3).

Put $x=1$ a third time, $7+2 = C(1+2)$; $\therefore$ $C = 3$, Lastly, make $x = -2$ in (3),

$$-14+2 = D(-2-1); \quad \therefore \quad D = 4.$$

Result $$\frac{1}{(x-1)^3} + \frac{2}{(x-1)^2} + \frac{3}{x-1} + \frac{4}{x+2}.$$

---

237 Thirdly.—When there is a quadratic factor of imaginary roots not repeated.

Ex.—Resolve $\frac{1}{(x^2+1)(x^2+x+1)}$ into partial fractions.

Here we must assume

$$\frac{1}{(x^2+1)(x^2+x+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+x+1};$$

x^2+1 and x^2+x+1 have no real factors, and are therefore retained as denominators. The requisite form of the numerators is seen by adding together two simple fractions, such as $\frac{a}{x+b} + \frac{c}{x+d}$.

Multiplying up, we have the equation

$$1 = (Ax+B)(x^2+x+1) + (Cx+D)(x^2+1) \quad \text{.............. (1).}$$

Let $$x^2+1 = 0; \quad \therefore \quad x^2 = 1.$$

Substitute this value of x^2 in (1) repeatedly; thus

$$1 = (Ax+B)x = Ax^2+Bx = -A+Bx;$$

or $$Bx-A-1 = 0.$$

Equate coefficients to zero; $\therefore \; B = 0,$

$$A = -1.$$

Again let $$x^2+x+1 = 0;$$

$$\therefore \quad x^2 = -x-1.$$

Substitute this value of x^2 repeatedly in (1); thus

$$1 = (Cx+D)(-x) = -Cx^2-Dx = Cx+C-Dx;$$

or $$(C-D)x+C-1 = 0.$$

Equate coefficients to zero; thus $C = 1,$

$$D = 1.$$

Hence $$\frac{1}{(x^2+1)(x^2+x+1)} = \frac{x+1}{x^2+x+1} - \frac{x}{x^2+1}.$$

---

238 Fourthly.—When there is a repeated quadratic factor of imaginary roots.

Ex.—Resolve $\frac{40x-103}{(x+1)^2(x^2-4x+8)^3}$ into partial fractions.

Assume

$$\frac{40x-103}{(x+1)^2(x^2-4x+8)} = \frac{Ax+B}{(x^2-4x+8)^3}+\frac{Cx+D}{(x^2-4x+8)^2}+\frac{Ex+F}{x^2-4x+8}$$

$$+\frac{G}{(x+1)^2}+\frac{H}{x+1};$$

$\therefore$

$$40x-103 = \{(Ax+B)+(Cx+D)(x^2-4x+8)+(Ex+F)(x^2-4x+8)^2\}(x+1)^2$$
$$+\{G+H(x+1)\}(x^2-4x+8)^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In the first place, to determine A and B, equate x^2-4x+8 to zero; thus $x^2 = 4x-8$.

Substitute this value of x^2 repeatedly in (1), as in the previous example, until the first power of x alone remains. The resulting equation is

$$40x-103 = (17A+6B)x-48A-7B.$$

Equating coefficients, we obtain two equations

$$\left.\begin{aligned}17A+6B &= 40\\ 48A+7B &= 103\end{aligned}\right\}, \text{ from which } \begin{aligned}A&=2\\ B&=1.\end{aligned}$$

Next, to determine C and D, substitute these values of A and B in (1); the equation will then be divisible by x^2-4x+8. Divide, and the resulting equation is

$$0 = 2x+13+\{Cx+D+(Ex+F)(x^2-4x+8)\}(x+1)^2$$
$$+\{G+H(x+1)\}(x^2-4x+8)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Equate x^2-4x+8 again to zero, and proceed exactly as before, when finding A and B.

Next, to determine E and F, substitute the values of C and D, last found in equation (2); divide, and proceed as before.

Lastly, G and H are determined by equating $x+1$ to zero successively, as in Example 2.

CONVERGENCY AND DIVERGENCY OF SERIES.

239 Let $a_1+a_2+a_3+$&c. be a series, and a_n, a_{n+1} any two consecutive terms. The following tests of convergency may be applied. The series will converge, if, after any fixed term—

(i.) The terms decrease and are alternately positive and negative.

(ii.) Or if $\frac{a_n}{a_{n+1}}$ is always *greater* than some quantity greater than unity.

(iii.) Or if $\frac{a_n}{a_{n+1}}$ is never less than the corresponding ratio in a known converging series.

(iv.) Or if $\left(\frac{na_n}{a_{n+1}} - n\right)$ is always *greater* than some quantity greater than unity. By 244, 234, & iii.

(v.) Or if $\left(\frac{na_n}{a_{n+1}} - n - 1\right) \log n$ is always *greater* than some quantity greater than unity.

---

240 The conditions of divergency are obviously the converse of rules (i.) to (v.)

---

241 The series $a_1 + a_2x + a_3x^2 + \&c.$ converges, if $\frac{a_{n+1}}{a_n}$ is always less than some quantity p, and x less than $\frac{1}{p}$.

By 239 (ii.)

---

242 To make the sum of the last series less than an assigned quantity p, make x less than $\frac{p}{p+k}$, k being the greatest coefficient.

General Theorem.

243 If $\phi(x)$ be positive for all positive integral values of x, and continually diminish as x increases, and if m be any positive integer, then the two series

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \ldots\ldots$$

$$\phi(1) + m\phi(m) + m^2\phi(m^2) + m^3\phi(m^3) + \ldots\ldots$$

are either both convergent or both divergent.

---

244 Application of this theorem. To ascertain whether the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots\ldots$$

is divergent or convergent when p is greater than unity.

Taking $m = 2$, the second series in (243) becomes

$$1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \&c. \ldots\ldots$$

a geometrical progression which converges; therefore the given series converges.

---

245 The series of which $\frac{1}{n(\log n)^p}$ is the general term is convergent if p be greater than unity, and divergent if p be not greater than unity. By (243), (244).

---

246 The series of which the general term is

$$\frac{1}{n\lambda(n)\,\lambda^2(n)\ldots\ldots\lambda^r(n)\,\{\lambda^{r+1}(n)\}^p}$$

where $\lambda(n)$ signifies $\log n$, $\lambda^2(n)$ signifies $\log\{\log(n)\}$, and so on, is convergent if p be greater than unity, and divergent if p be not greater than unity. By Induction, and by (243).

---

247 The series $a_1 + a_2 + \&c.$ is convergent if

$$na_n \log(n) \log^2(n) \ldots\ldots \log^r(n)\,\{\log^{r+1}(n)\}^p$$

is always finite for a value of p greater than unity; $\log^2(n)$ here signifying $\log(\log n)$, and so on.

[See Todhunter's Algebra, or Boole's Finite Differences.

EXPANSION OF A FRACTION.

---

248 A fractional expression such as $\frac{4x - 10x^2}{1 - 6x + 11x^2 - 6x^3}$ may be expanded in ascending powers of x in three different ways.

First, by dividing the numerator by the denominator in the ordinary way, or by Synthetic Division, as shewn in (29).

Secondly, by the method of Indeterminate Coefficients (232).

Thirdly, by Partial Fractions and the Binomial Theorem.

To expand by the method of Indeterminate Coefficients, proceed as follows:—

Assume $\dfrac{4x-10x^2}{1-6x+11x^2-6x^3} = A+Bx+Cx^2+Dx^3+Ex^4+\&c.\ldots$

$$\begin{array}{rrrrrrl} \therefore \quad 4x-10x^2 = & A+ & Bx+ & Cx^2+ & Dx^3+ & Ex^4+ & Fx^5+\ldots \\ & & -6Ax- & 6Bx^2- & 6Cx^3- & 6Dx^4- & 6Ex^5-\ldots \\ & & & +11Ax^2+ & 11Bx^3+ & 11Cx^4+ & 11Dx^5+\ldots \\ & & & & -6Ax^3- & 6Bx^4- & 6Cx^5-\ldots \end{array}$$

Equate coefficients of like powers of x, thus

$$\begin{array}{rll} A = & 0, & \\ B-6A = & 4, & \therefore\ B = 4; \\ C-6B+11A = & -10, & \therefore\ C = 14; \\ D-6C+11B-6A = & 0, & \therefore\ D = 40; \\ E-6D+11C-6B = & 0, & \therefore\ E = 110; \\ F-6E+11D-6C = & 0, & \therefore\ F = 304; \\ \ldots\ \ldots\ \ldots & \ldots\ \ldots & \ldots\ \ldots \end{array}$$

The formation of the same coefficients by synthetic division is now exhibited, in order that the connexion between the two processes may be clearly seen.

The division of $4x-10x^2$ by $1-6x+11x^2-6x^3$ is as follows:—

$$\begin{array}{r|l} & 0+4-10 \\ +6 & \qquad 24+84+240+660 \\ -11 & \qquad\quad -44-154-440-1210 \\ +6 & \qquad\qquad +24+84+240+660 \\ \hline & 0+4+14+40+110+304+\ldots\ldots\ldots\ldots \\ & A \quad B \quad C \quad D \quad E \quad F \end{array}$$

If we stop at the term $110x^4$, then the undivided remainder will be $304x^5-970x^6+660x^7$, and the complete result will be

$$4x+14x^2+40x^3+110x^4+\frac{304x^5-970x^6+660x^7}{1-6x+11x^2-6x^3}.$$

249 Here the concluding fraction may be regarded as the sum to infinity after four terms of the series, just as the original expression is considered to be the sum to infinity of the whole series.

250 If the general term be required, the method of expansion by partial fractions must be adopted. See (257), where the general term of the foregoing series is obtained.

RECURRING SERIES.

$a_0+a_1x+a_2x^2+a_3x^3+$&c. is a recurring series if the coefficients are connected by the relation

251 $$a_n = p_1a_{n-1}+p_2a_{n-2}+ \dots +p_ma_{n-m}.$$

The Scale of Relation is

252 $$1-p_1x-p_2x^2- \dots -p_mx^m.$$

The sum of n terms of the series is equal to

253 [The first m terms

$-p_1x$ (first $m-1$ terms $+$ the last term)

$-p_2x^2$ (first $m-2$ terms $+$ the last 2 terms)

$-p_3x^3$ (first $m-3$ terms $+$ the last 3 terms)

... ...

$-p_{m-1}x^{m-1}$ (first term $+$ the last $m-1$ terms)

$-p_mx^m$ (the last m terms)]$\div[1-p_1x-p_2x^2-\dots-p_mx^m]$.

254 If the series converges, and the sum to infinity is required, omit all "the last terms" from the formula.

255 EXAMPLE.—Required the Scale of Relation, the general term, and the apparent sum to infinity of the series

$$4x+14x^2+40x^3+110x^4+304x^5+854x^6+\dots$$

Observe that six arbitrary terms given are sufficient to determine a Scale of Relation of the form $1-px-qx^2-rx^3$, involving three constants p, q, r, for, by (251), we can write three equations to determine these constants; namely,

$$\left.\begin{aligned} 110 &= 40p+14q+4r \\ 304 &= 110p+40q+14r \\ 854 &= 304p+110q+40r \end{aligned}\right\} \text{The solution gives } p=6,\quad q=-11,\quad r=6.$$

Hence the Scale of Relation is $1-6x+11x^2-6x^3$.

The sum of the series without limit will be found from (254), by putting $p_1=6$, $p_2=-11$, $p_3=6$, $m=3$.

$$\begin{aligned} &\text{The first three terms} &&= 4x+14x^2+40x^3 \\ &-6x\times\text{the first two terms} &&= \quad -24x^2-84x^3 \\ &+11x\times\text{the first term} &&= \qquad\qquad +44x^3 \\ \hline & && \quad 4x-10x^2 \end{aligned}$$

$$\therefore \quad S = \frac{4x - 10x^2}{1 - 6x + 11x^2 - 6x^3};$$

the meaning of which is, that if this fraction be expanded in ascending powers of x, the first six terms will be those given in the question.

256 To obtain more terms of the series, we may use the Scale of Relation; thus the 7th term will be

$$(6 \times 854 - 11 \times 304 + 6 \times 110)\, x^7 = 2440x^7.$$

257 To find the general term, S must be decomposed into partial fractions; thus, by the method of (235),

$$\frac{4x - 10x^2}{1 - 6x + 11x^2 - 6x^3} = \frac{1}{1-3x} + \frac{2}{1-2x} - \frac{3}{1-x}.$$

By the Binomial Theorem (128),

$$\frac{1}{1-3x} = 1 + 3x + 3^2x^2 + \ldots\ldots + 3^n x^n,$$

$$\frac{2}{1-2x} = 2 + 2^2x + 2^3x^2 + \ldots\ldots + 2^{n+1}x^n,$$

$$-\frac{3}{1-x} = -3 - 3x - 3x^2 - \ldots\ldots - 3x^n.$$

Hence the general term involving x^n is

$$(3^n + 2^{n+1} - 3)\, x^n.$$

And by this formula we can write the "last terms" required in (253), and so obtain the sum of any finite number of terms of the given series. Also, by the same formula we can calculate the successive terms at the beginning of the series. In the present case this mode will be more expeditious than that of employing the Scale of Relation.

258 If, in decomposing S into partial fractions for the sake of obtaining the general term, a quadratic factor with imaginary roots should occur as a denominator, the same method must be pursued, for the imaginary quantities will disappear in the final result. In this case, however, it is more convenient to employ a general formula. Suppose the fraction which gives rise to the imaginary roots to be

$$\frac{L + Mx}{a + bx + x^2} = \frac{L + Mx}{(p - x)(q - x)},$$

p and q being the imaginary roots of $a + bx + x^2 = 0$.

Suppose $\quad p = \alpha + i\beta,$

$\quad q = \alpha - i\beta,$ where $i = \sqrt{(-1)}$.

If, now, the above fraction be resolved into two partial fractions in the ordinary way, and if these fractions be expanded separately by the Binomial Theorem, and that part of the general term furnished by these two expansions written out, still retaining p and q, and if the imaginary values of p and q be then substituted, it will be found that the factor will disappear, and that the result may be enunciated as follows.

259 The coefficient of x^{n-1} in the expansion of

$$\frac{L+Mx}{(\alpha^2+\beta^2)-2\alpha x+x^2}$$

will be

$$\frac{L}{\beta(\alpha^2+\beta^2)}\{n\alpha^{n-1}\beta-C(n,3)\,\alpha^{n-3}\beta^3+C(n,5)\,\alpha^{n-5}\beta^5-\dots\}$$

$$+\frac{M}{\beta(\alpha^2+\beta^2)^{n-1}}\{(n-1)\,\alpha^{n-2}\beta-C(n-1,3)\,\alpha^{n-4}\beta^3 +C(n-1,5)\,\alpha^{n-6}\beta^5-\dots\}.$$

260 With the aid of the known expansion of $\sin n\theta$ in Trigonometry, this formula for the n^{th} term may be reduced to

$$\sqrt{\frac{(L+M\alpha)^2+M^2\beta^2}{\beta^2(\alpha^2+\beta^2)^n}}\,.\sin(n\theta-\phi),$$

in which $\quad \theta=\tan^{-1}\dfrac{\beta}{\alpha},\qquad \phi=\tan^{-1}\dfrac{M\beta}{L+M\alpha}.$

If n be not greater than 100, $\sin(n\theta-\phi)$ may be obtained from the tables correct to about six places of decimals, and accordingly the n^{th} term of the expansion may be found with corresponding accuracy. As an example, the 100^{th} term in the expansion of $\dfrac{1+x}{5-2x+x^2}$ is readily found by this method to be $\dfrac{41824}{10^{41}}\,x^{99}$.

To determine whether a given Series is recurring or not.

261 If certain first terms only of the series be given, a scale of relation may be found which shall produce a recurring series whose first terms are those given. The method is exemplified in (255). The number of unknown coefficients p, q, r, &c. to be assumed for the scale of relation must be equal to half the number of the given terms of the series, if that number be even. If the number of given terms be odd, it may be made even by prefixing zero for the first term of the series.

262 Since this method may, however, produce zero values for one or more of the last coefficients in the scale of relation, it may be advisable in practice to determine a scale from the first two terms of the series, and if that scale does not produce the following terms, we may try a scale determined from the first four terms, and so on until the true scale is arrived at.

If an indefinite number of terms of the series be given, we may find whether it is recurring or not by a rule of Lagrange's.

263 Let the series be

$$S = A+Bx+Cx^2+Dx^3+\&c.$$

Divide unity by S as far as two terms of the quotient, which will be of the form $p+qx$, and write the remainder in the form $S'x^2$, S' being another indefinite series of the same form as S.

Next, divide S by S' as far as two terms of the quotient, and write the remainder in the form $S''x^2$.

Again, divide S' by S'', and proceed as before, and repeat this process until there is no remainder after one of the divisions. The series will then be proved to be a recurring series, and the order of the series, that is, the degree of the scale of relation, will be the same as the number of divisions which have been effected in the process.

EXAMPLE.—To determine whether the series 1, 3, 6, 10, 15, 21, 28, 36, 45, ... is recurring or not.

Introducing x, we may write

$$S = 1+3x+6x^2+10x^3+15x^4+21x^5+28x^6+36x^7+45x^8\ldots$$

Then we shall have $\frac{1}{S} = 1-3x+\ldots$ with a remainder

$$3x^2+8x^3+15x^4+24x^5+35x^6+\&c.$$

Therefore $\quad S' = 3+8x+15x^2+24x^3+35x^4+\&c.,$

$$\frac{S}{S'} = \frac{1}{3}+\frac{x}{9},$$

with a remainder $\frac{1}{9}(x^2+3x^3+6x^4+10x^5+\&c....)$.

Therefore we may take $S''=1+3x+6x^2+10x^3+\&c.$

Lastly $\frac{S'}{S''}=3-x$ without any remainder.

Consequently the series is a recurring series of the third order. It is, in fact, the expansion of $\frac{1}{1-3x+3x^2-x^3}$.

SUMMATION OF SERIES BY THE METHOD OF DIFFERENCES.

264 Rule.—*Form successive series of differences until a series of equal differences is obtained.* Let a, b, c, d, &c. be the first terms of the several series; then the n^{th} term of the given series is

265 $$a+(n-1)\,b+\frac{(n-1)(n-2)}{1\,.\,2}\,c+\frac{(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3}\,d+$$

The sum of n terms

266 $$=na+\frac{n\,(n-1)}{1\,.\,2}\,b+\frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,c+\&c.$$

[Proved by Induction.

Example:
a ... 1+ 5+15+35+70+126+...
b ... 4+10+20+35+56+...
c ... 6+10+15+21+...
d ... 4+5+6+...
e ... 1+1+...

The 100^{th} term of the first series

$$=1+99\,.\,4+\frac{99\,.\,98}{1\,.\,2}\,6+\frac{99\,.\,98\,.\,97}{1\,.\,2\,.\,3}\,4+\frac{99\,.\,98\,.\,97\,.\,96}{1\,.\,2\,.\,3\,.\,4}.$$

The sum of 100 terms

$$=100+\frac{100\,.\,99}{1\,.\,2}\,4+\frac{100\,.\,99\,.\,98}{1\,.\,2\,.\,3}\,6+\frac{100\,.\,99\,.\,98\,.\,97}{1\,.\,2\,.\,3\,.\,4}\,4+\frac{100\,.\,99\,.\,98\,.\,97\,.\,96}{1\,.\,2\,.\,3\,.\,4\,.\,5}.$$

267 To interpolate a term between two terms of a series by the method of differences.

Ex.—Given log 71, log 72, log 73, log 74, it is required to find log 72·54.
Form the series of differences from the given logarithms, as in (267),

	log 71	log 72	log 73	log 74
a ...	1·8512583	1·8573325	1·8633229	1·8692317
b ...	·0060742	·0059904	·0059088	
c ...	−·0000838	−·0000816		
d ...	−·0000022	considered to vanish.		

Log 72·54 must be regarded as an interpolated term, the number of its place being 2·54.

Therefore put 2·54 for n in formula (265).

Result log 72·54 = 1·8605777.

DIRECT FACTORIAL SERIES.

268 Ex.: $5.7.9+7.9.11+9.11.13+11.13.15+\ldots$

d = common difference of factors,

m = number of factors in each term,

n = number of terms,

a = first factor of first term $-d$.

$$n^{\text{th}} \text{ term} = (a+nd)(a+\overline{n+1}\,d)\ldots\ldots(a+\overline{n+m-1}\,d).$$

269 To find the sum of n terms. Rule.—*From the last term with the next highest factor take the first term with the next lowest factor, and divide by* $(m+1)\,d$. Proof, by Induction.

Thus the sum of 4 terms of the above series will be, putting $d=2$, $m=3$, $n=4$, $a=3$, $$S=\frac{11.13.15.17-3.5.7.9}{(3+1)\,2}$$

Proved either by Induction, or by the method of Indeterminate Coefficients.

INVERSE FACTORIAL SERIES.

270 Ex.: $\frac{1}{5.7.9}+\frac{1}{7.9.11}+\frac{1}{9.11.13}+\frac{1}{11.13.15}+\ldots$

Defining d, m, n, a as before, the

$$n^{\text{th}}\text{ term}=\frac{1}{(a+nd)\,(a+\overline{n+1}\,d)\ldots a+\overline{n+m-1}\,d}$$

271 To find the sum of n terms. Rule.—*From the first term wanting its last factor take the last term wanting its first factor, and divide by* $(m-1)\,d$. Proof, by Induction.

Thus the sum of 4 terms of the above series will be, putting $d=2$, $m=3$, $n=4$, $a=3$,

$$\frac{\frac{1}{5.7}-\frac{1}{13.15}}{(3-1)\,2}$$

Proved either by Induction, or by decomposing the terms, as in the following example.

272 Ex.: To sum the same series by decomposing the terms into partial fractions. Take the general term in the simple form

$$\frac{1}{(r-2)\,r\,(r+2)}$$

Resolve this into the three fractions

$$\frac{1}{8(r-2)}-\frac{1}{4r}+\frac{1}{8(r+2)}\text{ by (235).}$$

Substitute, 7, 9, 11, &c. successively for r, and the given series has for its equivalent the three series

$$\frac{1}{8}\left\{\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}\ \ldots\ldots\ldots\ldots\ +\frac{1}{2n+3}\right\}$$
$$+\frac{1}{8}\left\{-\frac{2}{7}-\frac{2}{9}-\frac{2}{11}-\frac{2}{13}-\ \ldots\ldots\ -\frac{2}{2n+3}-\frac{2}{2n+5}\right\}$$
$$+\frac{1}{8}\left\{\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\ldots\ldots+\frac{1}{2n+3}+\frac{1}{2n+5}+\frac{1}{2n+7}\right\}$$

and the sum of n terms is seen, by inspection, to be

$$\frac{1}{8}\left\{\frac{1}{5}-\frac{1}{7}-\frac{1}{2n+5}+\frac{1}{2n+7}\right\}=\frac{1}{4}\left\{\frac{1}{5.7}-\frac{1}{(2n+5)(2n+7)}\right\}$$

a result obtained at once by the rule in (271), taking $\frac{1}{5.7.9}$ for the first term, and $\frac{1}{(2n+3)(2n+5)(2n+7)}$ for the n^{th} or last term.

273 Analogous series may be reduced to the types in (268) and (270), or else the terms may be decomposed in the manner shewn in (272).

Ex.: $$\frac{1}{1.2.3}+\frac{4}{2.3.4}+\frac{7}{3.4.5}+\frac{10}{4.5.6}+\ldots\ldots$$

has for its general term

$$\frac{3n-2}{n(n+1)(n+2)}=-\frac{1}{n}+\frac{5}{n+1}-\frac{4}{n+2} \text{ by (235)},$$

and we may proceed as in (272) to find the sum of n terms.

The method of (272) includes the method known as 'Summation by Subtraction,' but it has the advantage of being more general and easier of application to complex series.

COMPOSITE FACTORIAL SERIES.

274 If the two series

$$(1-x)^{-5}=1+5x+\frac{5.6}{1.2}x^2+\frac{5.6.7}{1.2.3}x^3+\frac{5.6.7.8}{1.2.3.4}x^4+\ldots$$

$$(1-x)^{-3}=1+3x+\frac{3.4}{1.2}x^2+\frac{3.4.5}{1.2.3}x^3+\frac{3.4.5.6}{1.2.3.4}x^4+\ldots$$

be multiplied together, and the coefficient of x^4 in the product be equated to the coefficient of x^4 in the expansion of $(1-x)^{-8}$, we obtain as the result the sum of the composite series

$$5.6.7.8\times1.2+4.5.6.7\times2.3+3.4.5.6\times3.4$$
$$+2.3.4.5\times4.5+1.2.3.4\times5.6=\frac{\lfloor 4 \, \lfloor 11}{\lfloor 7 \, \lfloor 4}.$$

275 Generally, if the given series be

$$P_1Q_1+P_2Q_2+\ldots+P_{n-1}Q_{n-1} \quad \ldots\ldots\ldots\ldots\ldots (1),$$

where $\quad Q_r=r(r+1)(r+2)\ldots(r+q-1),$

and $\quad P_r=(n-r)(n-r+1)\ldots(n-r+p-1);$

the sum of $n-1$ terms will be

$$\frac{\lfloor p \, \lfloor q}{\lfloor p+q+1}\cdot\frac{\lfloor n+p+q-1}{\lfloor n-2}$$

MISCELLANEOUS SERIES.

276 *Sum of the powers of the terms of an Arithmetical Progression.*

$$1+2+3+\ldots+n = \frac{n(n+1)}{2} = S_1$$

$$1+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} = S_2$$

$$1+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = S_3$$

$$1+2^4+3^4+\ldots+n^4 = \frac{n(n+1)(2n+1)(3n^2+n-1)}{30} = S_4$$

By the method of Indeterminate Coefficients (234).

A general formula for the sum of the r^{th} powers of $1.2.3 \ldots n$, obtained in the same way, is

$$S_r = \frac{1}{r+1}\, n^{r+1} + \tfrac{1}{2}n^r + A_1 n^{r-1} + \ldots A_{r-1} n,$$

where A_1, A_2, &c. are determined by putting $p = 1, 2, 3,$ &c. successively in the equation

$$\frac{1}{2\,\underline{|p+2}}$$

$$= \frac{1}{\underline{|p+2}} + \frac{A_1}{r\,\underline{|p}} + \frac{A_2}{r(r-1)(p-1)} + \ldots \frac{A_p}{r(r-1)\ldots(r-p+1)}$$

277 $a^m+(a+d)^m+(a+2d)^m+\ldots+(a+nd)^m$

$$= (n+1)a^m + S_1 m a^{m-1}d + C(m,2)\,a^{m-2}d^2 + C(m,3)\,a^{m-3}d^3 + \&c.$$

By Bin. Theor. and (276).

278 *Summation of a series partly Arithmetical and partly Geometrical.*

EXAMPLE.—To find the sum of the series $1+3x+5x^2+$ to n terms.

Let $s = 1+3x+5x^2+7x^3+\ldots+(2n-1)\,x^{n-1}$

$\therefore\ sx = x+3x^2+5x^3+\ldots+(2n-3)\,x^{n-1}+(2n-1)\,x^n$

$\therefore$ by subtraction,

$$s\,(1-x) = 1+2x+2x^2+2x^3+\ldots+2x^{n-1}-(2n-1)\,x^n$$

$$= 1+2x\,\frac{1-x^{n-1}}{1-x}-(2n-1)\,x^n$$

$$\therefore\quad s = \frac{1-(2n-1)\,x^n}{1-x}+\frac{2x\,(1-x^{n-1})}{(1-x)^2}$$

---

279 A general formula for the sum of n terms of

$$a+(a+d)\,r+(a+2d)\,r^2+(a+3d)\,r^3+\&c.$$

is
$$S = \frac{a-(a+\overline{n-1}\,d)\,r^n}{1-r}+\frac{dr\,(1-r^{n-1})}{(1-r)^2}$$

Obtained as in (278).

RULE.—*Multiply by the ratio and subtract the resulting series.*

---

280 $$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots+x^{n-1}+\frac{x^n}{1-x}$$

281 $$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\ldots$$
$$\ldots+nx^{n+1}+\frac{(n+1)\,x^n-nx^{n+1}}{(1-x)^2}$$

---

282 $(n-1)\,x+(n-2)\,x^2+(n-3)\,x^3+\ldots+2x^{n-2}+x^{n-1}$

$$= \frac{(n-1)\,x-nx^2+x^{n+1}}{(1-x)^2}$$ By (253).

---

283 $$1+n+\frac{n\,(n-1)}{\lfloor 2}+\frac{n\,(n-1)\,(n-2)}{\lfloor 3}+\&c. = 2^n,$$

$$1-n+\frac{n\,(n-1)}{\lfloor 2}-\frac{n\,(n-1)\,(n-2)}{\lfloor 3}+\&c. = 0.$$

By making $x=1$ in (125).

284 The series

$$1-\frac{n-3}{2}+\frac{(n-4)(n-5)}{\lfloor 3}-\frac{(n-5)(n-6)(n-7)}{\lfloor 4}+\dots$$

$$\dots+(-1)^{r-1}\frac{(n-r-1)(n-r-2)\dots(n-2r+1)}{\lfloor r}$$

consists of $\frac{n}{2}$ or $\frac{n-1}{2}$ terms, and the sum is given by

$$S=\ \frac{3}{n} \text{ if } n \text{ be of the form } 6m+3,$$

$$S=\ 0 \text{ if } n \text{ be of the form } 6m\pm 1,$$

$$S=-\frac{1}{n} \text{ if } n \text{ be of the form } 6m,$$

$$S=\ \frac{2}{n} \text{ if } n \text{ be of the form } 6m\pm 2.$$

By (545), putting $p=x+y$, $q=xy$, and applying (546).

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285 The series $n^r-n(n-1)^r+\frac{n(n-1)}{\lfloor 2}(n-2)^r$

$$-\frac{n(n-1)(n-2)}{\lfloor 3}(n-3)^r+\&c.\dots$$

takes the values 0, $\lfloor n$, $\frac{1}{2}n\lfloor n+1$,

according as r is $<n$, $=n$, or $=n+1$.

Proved by expanding $(e^x-1)^n$, in two ways: first, by the Exponential Theorem and Multinomial; secondly, by the Bin. Th., and each term of the expansion by the Exponential. Equate the coefficients of x^r in the two results.

Other results by putting $r=n+2$, $n+3$, &c.

The series (285), when divided by $\lfloor r$, is, in fact, equal to the coefficient of x^r in the expansion of

$$\left\{x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots\right\}^n$$

286 By exactly the same process we may deduce from the function $\{e^x - e^{-x}\}^n$ the result that the series

$$n^r - n(n-2)^r + \frac{n(n-1)}{\underline{|2}}(n-4)^r - \&c.$$

takes the values 0 or $2^n \underline{|n}$, according as r is $< n$ or $= n$; this series, divided by $\underline{|r}$, being equal to the coefficient of x^r in the expansion of

$$2^n\left\{x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \dots\right\}^n$$

POLYGONAL NUMBERS.

287 The n^{th} term of the r^{th} order of polygonal numbers is equal to the sum of n terms of an Arith. Prog. whose first term is unity and common difference $r-2$; that is

$$= \frac{n}{2}\{2+(n-1)(r-2)\} = n + \tfrac{1}{2}n(n-1)(r-2).$$

288 The sum of n terms

$$= \frac{n(n+1)}{2} + \frac{n(n-1)(n+2)(r-2)}{6}$$

By resolving into two series.

Order.	n^{th} term.							
1	1	1	1	1	1	1	1	1
2	n	1	2	3	4	5	6	7
3	$\frac{1}{2}n(n-1)$	1	3	6	10	15	21	28
4	n^2	1	4	9	16	25	36	49
5	$\frac{1}{2}n(3n-1)$	1	5	12	22	35	51	70
6	$(2n-1)n$	1	6	15	28	45	66	91
...	... ...	...	...	...	...	...	...	...
r	$n+\frac{n(n-1)}{2}(r-2)$	1,	r,	$3+3(r-2)$,	$4+6(r-2)$,	$5+10(r-2)$, $6+15(r-2)$, &c.		

In practice—to form, for instance, the 6th order of polygonal numbers—write the first three terms by the formula, and form the rest by the method of differences.

Ex.:	1	6	15	28	45	66	91	120 ...	
		5	9	13	17	21	25	29 ...	
$[r-2=4]$			4	4	4	4	4	4 ...	

FIGURATE NUMBERS.

289 The n^{th} term of any order is the sum of n terms of the preceding order.

The n^{th} term of the r^{th} order is

$$\frac{n(n+1)\dots(n+r-2)}{\underline{|r-1}} = H(r, n-1) \qquad \text{By (98).}$$

290 The sum of n terms is

$$\frac{n(n+1)\dots(n+r-1)}{\underline{|r}} = H(r+1, n-1)$$

Order.	Figurate Numbers.	n^{th} term.
1	1, 1, 1, 1, 1, 1	1
2	1, 2, 3, 4, 5, 6	n
3	1, 3, 6, 10, 15, 21	$\frac{n(n+1)}{1.2}$
4	1, 4, 10, 20, 35, 56	$\frac{n(n+1)(n+2)}{1.2.3}$
5	1, 5, 15, 35, 70, 126	$\frac{n(n+1)(n+2)(n+3)}{1.2.3.4}$
6	1, 6, 21, 56, 126, 252	$\frac{n(n+1)(n+2)(n+3)(n+4)}{1.2.3.4.5}$

HYPERGEOMETRICAL SERIES.

---

291 $$1+\frac{\alpha.\beta}{1.\delta}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2$$
$$+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\&c.$$

is convergent if x is < 1,
and divergent if x is > 1; (239 ii.)
and if $x = 1$, the series is
convergent if $\gamma-\alpha-\beta$ is positive,
divergent if $\gamma-\alpha-\beta$ is negative, (239 iv.)
and divergent if $\gamma-\alpha-\beta$ is zero. (239 v.)

Let the hypergeometrical series (291) be denoted by $F(\alpha, \beta, \gamma)$; then, the series being convergent, it is shewn by induction that

292 $$\frac{F(\alpha,\beta+1,\gamma+1)}{F(\alpha,\beta,\gamma)}=\cfrac{1}{1-\cfrac{k_1}{1-\cfrac{k_2}{1-\&c....}}}$$ concluding with $$1-\frac{k_{2r-1}}{1-k_{2r}z_{2r}}$$

where k_1, k_2, k_3, &c., with z_{2r}, are given by the formulæ

$$k_{2r-1}=\frac{(\alpha+r-1)(\gamma+r-1-\beta)x}{(\gamma+2r-2)(\gamma+2r-1)}$$

$$k_{2r}=\frac{(\beta+r)(\gamma+r-\alpha)x}{(\gamma+2r-1)(\gamma+2r)}$$

$$z_{2r}=\frac{F(\alpha+r,\beta+r+1,\gamma+2r+1)}{F(\alpha+r,\beta+r,\gamma+2r)}$$

The continued fraction may be concluded at any point with $k_{2r}z_{2r}$. When r is infinite, $z_{2r}=1$ and the continued fraction is infinite.

293 Let

$$f(\gamma) \equiv 1+\frac{x^2}{1.\gamma}+\frac{x^4}{1.2.\gamma(\gamma+1)}+\frac{x^6}{1.2.3.\gamma(\gamma+1)(\gamma+2)}+\&c.$$

the result of substituting $\frac{x^2}{\alpha\beta}$ for x in (291), and making $\beta=\alpha=\infty$. Then, by last, or independently by induction,

$$\frac{f(\gamma+1)}{f(\gamma)}=\frac{1}{1+}\,\frac{p_1}{1+}\,\frac{p_2}{1+}\dots+\frac{p_m}{1+}\&c.$$

$$\text{with } p_m=\frac{x^2}{(\gamma+m-1)(\gamma+m)}$$

294 In this result put $\gamma=\frac{1}{2}$ and $\frac{y}{2}$ for x, and we obtain by Exp. Th. (150),

$$\frac{e^y-e^{-y}}{e^y+e^{-y}}=\frac{y}{1+}\,\frac{y^2}{3+}\,\frac{y^2}{5+}\&c. \quad \text{the } r^{\text{th}} \text{ component being } \frac{y^2}{2r-1}$$

Or the continued fraction may be formed by ordinary division of one series by the other.

295 $e^{\frac{m}{n}}$ is incommensurable, m and n being integers. From the last and (174), by putting $x=\frac{m}{n}$.

INTEREST.

If r be the Interest on £1 for 1 year,
n the number of years,
P the Principal,
A the amount in n years. Then

296 At Simple Interest $A=P(1+nr)$.

297 At Compound Interest $A=P(1+r)^n$. By (84)

298 But if the payments of Interest be made q times a year ... ... $A = P\left(1+\frac{r}{q}\right)^{nq}$

If A be an amount due in n years time, and P the present worth of A. Then

299 At Simple Interest $P = \frac{A}{1+nr}$ By (296)

300 At Compound Interest $P = \frac{A}{(1+r)^n}$ By (297)

301 Discount $= A-P$.

ANNUITIES.

302 The amount of an Annuity of £1 in n years, at Simple Interest... $= n + \frac{n(n-1)}{2}r$ By (82)

303 Present value of same $= \frac{n+\frac{1}{2}n(n-1)r}{1+nr}$ By (299)

304 Amount at Compound Interest ... ... $= \frac{(1+r)^n-1}{(1+r)-1}$ By (85)

Present worth of same $= \frac{1-(1+r)^{-n}}{(1+r)-1}$ By (300)

305 Amount when the payments of Interest are made q times per annum ... ... $= \frac{\left(1+\frac{r}{q}\right)^{nq}-1}{\left(1+\frac{r}{q}\right)^{q}-1}$ By (298)

Present value of same $= \frac{1-\left(1+\frac{r}{q}\right)^{-nq}}{\left(1+\frac{r}{q}\right)^{q}-1}$

306 Amount when the payments of the Annuity are made m times per annum ... ... $= \dfrac{(1+r)^n-1}{m\{(1+r)^{\frac{1}{m}}-1\}}$

Present value of same $= \dfrac{1-(1+r)^{-n}}{m\{(1+r)^{\frac{1}{m}}-1\}}$

307 Amount when the Interest is paid q times and the Annuity m times per annum ... $= \dfrac{\left(1+\frac{r}{q}\right)^{nq}-1}{m\left\{\left(1+\frac{r}{q}\right)^{\frac{q}{m}}-1\right\}}$

Present value of same $= \dfrac{1-\left(1+\frac{r}{q}\right)^{-nq}}{m\left\{\left(1+\frac{r}{q}\right)^{\frac{q}{m}}-1\right\}}$

PROBABILITIES.

309 If all the ways in which an event can happen be m in number, all being equally likely to occur, and if in n of these m ways the event would happen under certain restrictive conditions; then the probability of the restricted event happening is equal to $n \div m$.

Thus, if the letters of the alphabet be chosen at random, any letter being equally likely to be taken, the probability of a vowel being selected is equal to $\frac{5}{26}$. The number of unrestricted cases here is 26, and the number of restricted ones 5.

310 If, however, all the m events are not equally probable, they may be divided into groups of equally probable cases. The probability of the restricted event happening in each group separately must be calculated, and the sum of these probabilities will be the total probability of the restricted event happening at all.

EXAMPLE.—There are three bags A, B, and C.

A contains 2 white and 3 black balls
B „ 3 „ 4 „
C „ 4 „ 5 „

A bag is taken at random and a ball drawn from it. Required the probability of the ball being white.

Here the probability of the bag A being chosen $= \frac{1}{3}$, and the subsequent probability of a white ball being drawn $= \frac{2}{5}$.

Therefore the probability of a white ball being drawn from A

$$= \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$$

Similarly the probability of a white ball being drawn from B

$$= \frac{1}{3} \times \frac{3}{7} = \frac{1}{7}$$

And the probability of a white ball being drawn from C

$$= \frac{1}{3} \times \frac{4}{9} = \frac{4}{27}$$

Therefore the total probability of a white ball being drawn

$$= \frac{2}{15} + \frac{1}{7} + \frac{4}{27} = \frac{401}{945}$$

If a be the number of ways in which an event can happen, and b the number of ways in which it can fail; then the

311 Probability of the event happening $= \dfrac{a}{a+b}$

312 Probability of the event failing $= \dfrac{b}{a+b}$

Thus Certainty $= 1$.

If p, p' be the respective probabilities of two *independent* events; then

313 Probability of both happening $= pp'$.

314 „ of not *both* happening $= 1-pp'$.

315 „ of one happening and one failing

$$= p+p'-2pp'.$$

316 „ of both failing $= (1-p)(1-p')$.

If the probability of an event happening in one trial be p, and the probability of its failing, q; then

317 Probability of the event happening r times in n trials

$$= C(n, r)\, p^r q^{n-r}.$$

318 Probability of the event failing r times in n trials

$$= C(n, r)\, p^{n-r} q^r.$$

[By induction.

319 Probability of the event happening *at least* r times in n trials = the sum of the *first* $n-r+1$ terms in the expansion of $(p+q)^n$.

320 Probability of the event failing *at least* r times in n trials = the sum of the *last* $n-r+1$ terms in the same expansion.

321 The number of trials in which the probability of the same event happening amounts to p'

$$= \frac{\log(1-p')}{\log(1-p)}$$

From the equation $(1-p)^x = 1-p'$.

322 Definition.—When a sum of money is to be received if a certain event happens, that sum multiplied into the probability of the event is termed the expectation.

Example.—If three coins be taken at random from a bag containing one sovereign, four half-crowns, and five shillings, the expectation will be the sum of the expectations founded upon each way of drawing three coins. But this is also equal to the average value of three coins out of the ten; that is, $\frac{3}{10}$ths of 45 shillings, or 13*s*. 6*d*.

323 The Probability that, after r chance selections of the numbers 0, 1, 2, 3 ... n, the sum of the numbers drawn will be s, is equal to the coefficient of x^s in the expansion of

$$(x^0+x^1+x^2+\ldots+x^n)^r \div (n+1)^r.$$

324 The probability of the existence of a certain cause of an observed event out of several known causes, one of which *must* have produced the event, is proportional to the *a priori* probability of the cause existing multiplied by the probability of the event happening from it if it does exist.

Thus, if the *a priori* probabilities of the causes be P_1, P_2, ... &c., and the corresponding probabilities of the event happening from those causes Q_1, Q_2 ... &c.; then, the probability of the r^{th} cause having produced the event is

$$\frac{P_r Q_r}{\Sigma(PQ)}$$

325 If P'_1, P'_2 ... &c. be the *a priori* probabilities of a second event happening from the same causes respectively; then, *after* the first event has happened, the probability of the second happening is $\dfrac{\Sigma(PQP')}{\Sigma(PQ)}$

For this is the sum of such probabilities as $\dfrac{P_r Q_r P'_r}{\Sigma(PQ)}$, which is the probability of the r^{th} cause existing multiplied by the probability of the second event happening from it.

Ex. 1.—Suppose there are

4 vases containing each 5 white and 6 black balls,
2 vases containing each 3 white and 5 black balls,
and 1 vase containing 2 white and 1 black ball.

A white ball has been drawn, and the probability that it came from the group of 2 vases is required.

Here $P_1 = \frac{4}{7}$, $P_2 = \frac{2}{7}$, $P_3 = \frac{1}{7}$;

$Q_1 = \frac{5}{11}$, $Q_2 = \frac{3}{8}$, $Q_3 = \frac{2}{3}$.

Therefore, by (324), the probability required is

$$P_2 = \frac{\frac{2.3}{7.8}}{\frac{4.5}{7.11} + \frac{2.3}{7.8} + \frac{1.2}{7.3}} = \frac{99}{427}$$

Ex. 2.—After the white ball has been drawn and replaced, a ball is drawn again; required the probability of the ball being black.

Here $P_1' = \frac{6}{11}, \quad P_2' = \frac{5}{8}, \quad P_3' = \frac{1}{3}.$

The probability, by (325), will be

$$\frac{\frac{4.5.6}{7.11.11}+\frac{2.3.5}{7.8.8}+\frac{1.2.1}{7.3.3}}{\frac{4.5}{7.11}+\frac{2.3}{7.8}+\frac{1.2}{7.3}} = \frac{58639}{112728}$$

If the probability of the second ball being white is required, $P_1P_2P_3$ must be employed instead of $P_1'P_2'P_3'$.

326 The probability of one event *at least* happening out of a number of events whose respective probabilities are a, b, c, &c., is

$$P_1 - P_2 + P_3 - P_4 + \&c.$$

where P_1 is the probability of 1 event happening,
P_2 ,, ,, 2 ,,

and so on. For, by (316), the probability is

$$1-(1-a)(1-b)(1-c)\ldots = \Sigma a - \Sigma ab + \Sigma abc -, \ldots$$

327 The probability of the occurrence of r assigned events and no more out of n events is

$$Q_r - Q_{r+1} + Q_{r+2} - Q_{r+3} + \&c.,$$

where Q_r is the probability of the r assigned events; Q_{r+1} the probability of $r+1$ events including the r assigned events.

For if a, b, c ... be the probabilities of the r events, and a', b', c' ... the probabilities of the excluded events, the required probability will be

$$abc \ldots (1-a')(1-b')(1-c')\ldots$$
$$= abc \ldots (1 - \Sigma a' + \Sigma a'b' - \Sigma a'b'c' + \ldots)$$

328 The probability of *any* r events happening and no more is

$$\Sigma Q_r - \Sigma Q_{r+1} + \Sigma Q_{r+2} - \&c.$$

NOTE.—If $a=b=c=\&c.$; then $\Sigma Q_r = C(n, r)\, Q_r$, &c.

INEQUALITIES.

330 $\frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}$ lies between the greatest and least of the fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots \frac{a_n}{b_n}$, the denominators being all of the same sign.

PROOF.—Let k be the greatest of the fractions, and $\frac{a_r}{b_r}$ any other; then $a_r < kb_r$. Substitute in this way for each a. Similarly if k be the least fraction.

331 $$\frac{a+b}{2} > \sqrt{ab}.$$

332 $$\frac{a_1+a_2+\ldots+a_n}{n} > \sqrt[n]{a_1 a_2 \ldots a_n};$$

or, Arithmetic mean > Geometric mean.

PROOF.—Substitute both for the greatest and least factors their Arithmetic mean. The product is thus *increased* in value. Repeat the process indefinitely. The limiting value of the G. M. is the A. M. of the quantities.

333 $$\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m$$

excepting when m is a positive proper fraction.

PROOF: $a^m+b^m = \left(\frac{a+b}{2}\right)^m \{(1+x)^m+(1-x)^m\}$

where $x = \frac{a-b}{a+b}$. Employ Bin. Th.

334 $$\frac{a_1^m+a_2^m+\ldots+a_n^m}{n} > \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m$$

excepting when m is a positive proper fraction, Otherwise.—*The Arithmetic mean of the m^{th} powers is greater than the m^{th} power of the Arithmetic mean, excepting when m is a positive proper fraction.*

Proof.—Similar to (332). Substitute for the greatest and least on the left side, employing (333).

---

336 If x and m are positive, and x and mx less than unity; then $$(1+x)^{-m} > 1-mx. \qquad (125, 240).$$

---

337 If x, m, and n are positive, and n greater than m; then, by taking x small enough, we can make

$$1+nx > (1+x)^m.$$

For x may be diminished until $1+nx$ is $>$ $(1-mx)^{-1}$, and this is $>$ $(1+x)^m$, by *last*.

---

338 If x be positive, $\log(1+x) < x$ (150).

If x be positive and > 1, $\log(1+x) > x - \frac{x^2}{2}$ (155, 240).

If x be positive and < 1, $\log\frac{1}{1-x} > x$ (156).

---

339 When n becomes infinite in the two expressions

$$\frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} \text{ and } \frac{3.5.7 \ldots (2n+1)}{2.4.6 \ldots 2n}$$

the first vanishes, the second becomes infinite, and their product lies between $\frac{1}{2}$ and 1.

Shewn by adding 1 to each factor (see 73), and multiplying the result by the original fraction.

---

340 If m be $> n$, and $n > a$,

$$\left(\frac{m+a}{m-a}\right)^m \text{ is } < \left(\frac{n+a}{n-a}\right)^n$$

341 If a, b be positive quantities,

$$a^a b^b \text{ is} > \left(\frac{a+b}{2}\right)^{a+b}$$

Similarly $$a^a b^b c^c > \left(\frac{a+b+c}{3}\right)^{a+b+c}$$

These and similar theorems may be proved by taking logarithms of each side, and employing the Expon. Th. (158), &c.

SCALES OF NOTATION.

342 If N be a whole number of $n+1$ digits, and r the radix of the scale, $N = p_n r^n + p_{n-1} r^{n-1} + p_{n-2} r^{n-2} + \dots + p_1 r + p_0$, where $p_n, p_{n-1}, \dots p_0$ are the digits.

343 Similarly a radix-fraction will be expressed by

$$\frac{p_1}{r} + \frac{p_2}{r^2} + \frac{p_3}{r^3} + \&c.$$

where p_1, p_2, &c. are the digits.

EXAMPLES: 3426 in the scale of 7 $= 3.7^3 + 4.7^2 + 2.7 + 6$

·1045 in the same scale $= \frac{1}{7} + \frac{0}{7^2} + \frac{4}{7^3} + \frac{5}{7^4}$

344 Ex.—To transform 34268 from the scale of 5 to the scale of 11.

RULE.—*Divide successively by the new radix.*

```
11 | 34268
  11 | 1343 — t
    11 | 40 — 3
          1 — 9
```

Result 193t, **in which** t **stands for 10.**

345 Ex.—To transform $\cdot t0e1$ from the scale of 12 to that of 7, e standing for 11, and t for 10.

Rule.—*Multiply successively by the new radix.*

$$\begin{array}{r} \cdot t0e1 \\ 7 \\ \hline 5\cdot t657 \\ 7 \\ \hline 6\cdot 19.1 \\ 7 \\ \hline 1\cdot 0497 \\ 7 \\ \hline 0\cdot 2971 \end{array} \qquad \text{Result } \cdot 5610 \ldots$$

346 Ex.—In what scale does $2t7$ represent the number 475 in the scale of ten?

Solve the equation $2r^2+10r+7 = 475$. [178.

Result $r = 13$.

347 The sum of the digits of any number divided by $r-1$ leaves the same remainder as the number itself divided by $r-1$; r being the radix of the scale. (401).

348 The difference between the sums of the digits in the even and odd places divided by $r+1$ leaves the same remainder as the number itself when divided by $r+1$.

THEORY OF NUMBERS.

349 If a is prime to b, $\frac{a}{b}$ is in its lowest terms.

Proof.—Let $\frac{a}{b} = \frac{a_1}{b_1}$, a fraction in lower terms.

Divide a by a_1, remainder a_2 quotient q_1,

b by b_1, remainder b_2 quotient q_1;

and so on, as in finding the H. C. F. of a and a_1, and of b and b_1 (see 30). Let a_n and b_n be the highest common factors thus determined.

Then, because $\frac{a}{b} = \frac{a_1}{b_1} \quad \therefore \quad \frac{a}{b} = \frac{a - q_1 a_1}{b - q_1 b_1} = \frac{a_2}{b_2}$ (70);

and so on; thus $\frac{a}{b} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \&c. \ldots\ldots = \frac{a_n}{b_n}$

Therefore a and b are equimultiples of a_n and b_n; that is, a is not prime to b if any fraction exists in lower terms.

---

350 If a is prime to b, and $\frac{a'}{b'} = \frac{a}{b}$; then a' and b' are equimultiples of a and b.

PROOF.—Let $\frac{a'}{b'}$ reduced to its lowest terms be $\frac{p}{q}$. Then $\frac{p}{q} = \frac{a}{b}$, and since p is now prime to q, and a prime to b; it follows, by 349, that $\frac{p}{q}$ is neither greater nor less than $\frac{a}{b}$; that is, it is equal to it. Therefore, &c.

---

351 If ab is divisible by c, and a is not; then b must be.

PROOF.—Let $\frac{ab}{c} = q \quad \therefore \quad \frac{a}{c} = \frac{q}{b}$

But a is prime to c; therefore, by last, b is a multiple of c.

352 If a and b be each of them prime to c, ab is prime to c. By last.

---

353 If $abcd \ldots$ is divisible by a prime, one at least of the factors a, b, c, &c. must also be divisible by it.

Or, if p be prime to all but one of the factors, that factor is divisible by p. (351).

354 Therefore, if a^n is divisible by p, p cannot be prime to a; and if p be a prime it must divide a.

---

355 If a is prime to b, any power of a is prime to any power of b.

Also, if a, b, c, &c. are prime to each other, the product of any of their powers is prime to any other product of their powers.

---

356 No expression with integral coefficients such as $A+Bx+Cx^2+\ldots$ can represent primes only.

PROOF.—For it is divisible by x if $A=0$; and if not, it is divisible by A, when $x=A$.

---

357 The number of primes is infinite.

PROOF.—Suppose p the greatest prime. Then the product of all primes up to p plus unity is either a prime, or divisible by a prime greater than p.

---

358 If a be prime to b, and the quantities a, $2a$, $3a$, ... $(b-1)\,a$ be divided by b, the remainders will be different.

PROOF.—Assume $ma-nb=m'a-n'b$, m and n being less than b, $\therefore\ \frac{a}{b}=\frac{n-n'}{m-m'}$. Then by (350).

---

359 A number can be resolved into prime factors in one way only. By (353).

360 To resolve 5040 into its prime factors.

RULE.—*Divide by the prime numbers successively.*

$$\begin{array}{r|l} 2\times5 & 5040 \\ 2 & 504 \\ 2 & 252 \\ 2 & 126 \\ 7 & 63 \\ 3 & 9 \\ & 3 \end{array}$$

Thus $5040 = 2^4.3^2.5.7.$

---

361 Required the least multiplier of 4704 which will make the product a perfect fourth power.

By (196), $4704 = 2^5.3.7^2.$

Then $2^5.3^1.7^2\times2^3.3^3.7^2 = 2^8.3^4.7^4 = 84^4.$

The indices 8, 4, 4 being the least multiples of 4 which are not less than 5, 1, 2 respectively.

Thus $2^3.3^3.7^2 = 3528$ is the multiplier required.

362 All numbers are of one of the forms $2n$ or $2n+1$

,,	,,	$2n$ or $2n-1$
,,	,,	$3n$ or $3n\pm1$
,,	,,	$4n$ or $4n\pm1$ or $4n+2$
,,	,,	$4n$ or $4n\pm1$ or $4n-2$
,,	,,	$5n$ or $5n\pm1$ or $5n\pm2$

and so on.

363 All square numbers are of the form $5n$ or $5n\pm1$.

Proved by squaring the forms $5n$, $5n\pm1$, $5n\pm2$, which comprehend all numbers whatever.

364 All cube numbers are of the form $7n$ or $7n\pm1$.
And similarly for other powers.

---

365 The highest power of a prime p, which is contained in the product $\lfloor m$, is the sum of the integral parts of

$$\frac{m}{p},\ \frac{m}{p^2},\ \frac{m}{p^3},\ \&c.$$

For there are $\frac{m}{p}$ factors in $\lfloor m$ which p will divide; $\frac{m}{p^2}$ which it will divide a second time; and so on. The successive divisions are equivalent to dividing by

$$p^{\frac{m}{p}}\,.\,p^{\frac{m}{p^2}}\ \ldots\ \&c. = p^{\frac{m}{p}+\frac{m}{p^2}+}$$

Example.—The highest power of 3 which will divide $\lfloor 29$. Here the factors 3, 6, 9, 12, 15, 18, 21, 24, 27 can be divided by 3. Their number is $\frac{29}{3} = 9$ (the integral part).

The factors 9, 18, 27 can be divided a second time. Their number is $\frac{29}{3^2} = 3$ (the integral part).

One factor, 27, is divisible a third time. $\frac{29}{3^3} = 1$ (integral part).

$9+3+1 = 13$; that is, 3^{13} is the highest power of 3 which will divide $\lfloor 29$.

---

366 The product of any r consecutive integers is divisible by $\lfloor r$.

Proof: $\dfrac{n(n-1)\ldots(n-r+1)}{\lfloor r}$ is necessarily an integer, by (96).

367 If n be a prime, every coefficient in the expansion of $(a+b)^n$, except the first and last, is divisible by n. By last.

368 If n be a prime the coefficient of every term in the expansion of $(a+b+c\,...)^n$, except a^n, b^n, &c., is divisible by n.

By last. Put β for $(b+c+\,...)$.

---

369 *Fermat's Theorem.*—If p be a prime, and N prime to p; then $N^{p-1}-1$ is divisible by p.

PROOF: $N^p = (1+1+...)^p = N+Mp$. By last.

370 If p be any number, and if $1, a, b, c, ... (p-1)$ be all the numbers less than, and prime to p; and if n be their number, and x any one of them; then x^n-1 is divisible by p.

PROOF.—If $x, ax, bx\, ...\, (p-1)\,x$ be divided by p, the remainders will be all different and prime to p (as in 358); therefore the remainders will be $1, a, b, c\, ...\, (p-1)$; therefore the product $x^n abc\, ...\, (p-1) = abc\, ...\, (p-1)+Mp$.

---

371 *Wilson's Theorem.*—If p be a prime, and only then, $1+\underline{|p-1}$ is divisible by p.

Put $p-1$ for r and n in (285), and apply Fermat's Theorem to each term.

372 If p be a prime $= 2n+1$, then $(\underline{|n})^2+(-1)^n$ is divisible by p.

PROOF.—By multiplying together equi-distant factors of $\underline{|p-1}$ in Wilson's Theorem, and putting $2n+1$ for p.

---

373 Let $N = a^p b^q c^r\, ...$ in prime factors, the number of integers including 1, which are less than N and prime to it, is

$$N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)...$$

PROOF.—The number of integers prime to N contained in a^p is $a^p - \dfrac{a^p}{a}$. Similarly in b^q, c^r, &c. Take the product of these.

Also the number of integers less than and prime to $(N \times M \times \&c.)$ is the product of the corresponding numbers for N, M, &c. separately.

374 The number of divisors of N, including 1 and N itself, is $= (p+1)(q+1)(r+1)\ldots.$ For it is equal to the number of terms in the product

$$(1+a+\ldots+a^p)(1+b+\ldots+b^q)(1+c+\ldots+c^r)\ldots\ \&c.$$

375 The number of ways of resolving N into two factors is half the number of its divisors (374). If the number be a square the two equal factors must, in this case, be reckoned as two divisors.

376 If the factors of each pair are to be prime to each other, put p, q, r, &c. each equal to one.

377 The sum of the divisors of N is

$$\frac{a^{p+1}-1}{a-1} \cdot \frac{b^{q+1}-1}{b-1} \cdot \frac{c^{r+1}-1}{c-1} \ldots$$

Proof.—By the product in (374), and by (85).

378 If p be a prime, then the $p-1^{\text{th}}$ power of any number is of the form mp or $mp+1$. By Fermat's Theorem (369).

Ex.—The 12^{th} power of any number is of the form $13m$ or $13m+1$.

379 To find all the divisors of a number; for instance, of 504.

I.	II.						
		1					
504	2	2					
252	2	4					
126	2	8					
63	3	3	6	12	24		
21	3	9	18	36	72		
7	7	7	14	28	56	21	42
		84	168	63	126	252	504

Explanation.—Resolve 504 into its prime factors, placing them in column II.

The divisors of 504 are now formed from the numbers in column II., and placed to the right of that column in the following manner:—

Place the divisor 1 to the right of column II., and follow this rule—*Multiply in order all the divisors which are written down by the next number in column II., which has not already been used as a multiplier: place the first new divisor so obtained and all the following products in order to the right of column II.*

380 S_r the sum of the r^{th} powers of the first n natural numbers is divisible by $2n+1$.

PROOF: $$x(x^2-1^2)(x^2-2^2)\dots(x^2-n^2)$$

constitutes $2n+1$ factors divisible by $2n+1$, by (366). Multiply out, rejecting x, which is to be less than $2n+1$. Thus, using (372),

$$x^{2n}-S_1x^{2n-2}+S_2x^{2n-4}-\dots S_{n-1}x^2+(-1)^n(\underline{|n})^2=M(2n+1).$$

Put 1, 2, 3 ... $(n-1)$ in succession for x, and the solution of the $(n-1)$ equations is of the form $S_r=M(2n+1)$.

THEORY OF EQUATIONS.

FACTORS OF AN EQUATION.

General form of a rational integral equation of the n^{th} degree.

400 $$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0.$$

The left side will be designated $f(x)$ in the following summary.

401 If $f(x)$ be divided by $x-a$, the remainder will be $f(a)$. By assuming $f(x) = P(x-a) + R$.

402 If a be a root of the equation $f(x) = 0$, then $f(a) = 0$.

403 To compute $f(a)$ numerically; *divide $f(x)$ by $x-a$, and the remainder will be $f(a)$.* [401

404 Example.—To find the value of $4x^6 - 3x^5 + 12x^4 - x^2 + 10$ when $x = 2$.

$$\begin{array}{r|l} & 4-3+12\ +0\ -1\ \ +0\ +10 \\ 2 & \quad 8+10+44+88+174+348 \\ \hline & 4+5+22+44+87+174+358 \end{array}$$

Thus $f(2) = 358$.

If $a, b, c \dots k$ be the roots of the equation $f(x) = 0$; then, by (401) and (402),

405 $$f(x) = p_0(x-a)(x-b)(x-c)\dots(x-k).$$

By multiplying out the last equation, and equating coefficients with equation (400), considering $p_0 = 1$, the following results are obtained:—

406 $-p_1 =$ the sum of all the roots of $f(x)$.

$p_2 =$ the sum of the products of the roots taken two at a time.

$-p_3 =$ the sum of the products of the roots taken three at a time.

... ...

... ...

$(-1)^r p_r =$ the sum of the products of the roots taken r at a time.

... ...

$(-1)^n p_n =$ product of all the roots.

407 The number of roots of $f(x)$ is equal to the degree of the equation.

408 Imaginary roots must occur in pairs of the form

$$\alpha+\beta\sqrt{-1}, \quad \alpha-\beta\sqrt{-1}.$$

The quadratic factor corresponding to these roots will then have real coefficients; for it will be

$$x^2-2\alpha x+\alpha^2+\beta^2. \qquad [405, 226$$

409 If $f(x)$ be of an odd degree, it has at least one real root of the opposite sign to p_n.

Thus $x^7-1=0$ has *at least* one positive root.

410 If $f(x)$ be of an even degree, and p_n negative, there is *at least* one positive and one negative root.

Thus x^4-1 has $+1$ and -1 for roots.

411 If several terms at the beginning of the equation are of one sign, and all the rest of another, there is one, and only one, positive root.

Thus $x^5+2x^4+3x^3+x^2-5x-4=0$ has only one positive root.

412 If all the terms are positive there is no positive root.

413 If all the terms of an even order are of one sign, and all the rest are of another sign, there is no negative root.

414 Thus $x^4-x^3+x^2-x+1=0$ has no negative root.

415 If all the indices are even, and all the terms of the same sign, there is no real root; and if all the indices are odd, and all the terms of the same sign, there is no real root but zero.

Thus $x^4+x^2+1=0$ has no real root, and $x^5+x^3+x=0$ has no real root but zero. In this last equation there is no absolute term, because such a term would involve the zero power of x, which is even, and by hypothesis is wanting.

DESCARTES' RULE OF SIGNS.

416 In the following theorems every two adjacent terms in $f(x)$, which have the same signs, count as one "continuation of sign"; and every two adjacent terms, with different signs, count as one change of sign.

417 $f(x)$, multiplied by $(x-a)$, has an *odd* number of changes of sign thereby introduced, and *one at least.*

418 $f(x)$ cannot have more positive roots than changes of sign, or more negative roots than continuations of sign.

419 When all the roots of $f(x)$ are real, the number of positive roots is equal to the number of changes of sign in $f(x)$; and the number of negative roots is equal to the number of changes of sign in $f(-x)$.

420 Thus, it being known that the roots of the equation

$$x^4-10x^3+35x^2-50x+24=0$$

are all real; the number of positive roots will be equal to the number of changes of sign, which is four. Also $f(-x)=x^4+10x^3+35x^2+50x+24=0$, and since there is no change of sign, there is consequently, by the rule, no negative root.

421 If the degree of $f(x)$ exceeds the number of changes of sign in $f(x)$ and $f(-x)$ together, by μ, there are at least μ imaginary roots.

422 If, between two terms in $f(x)$ of the same sign, there be an odd number of consecutive terms wanting, then there must be at least one more than that number of imaginary roots; and if the missing terms lie between terms of different sign, there is at least one less than the same number of imaginary roots.

Thus, in the cubic equation $x^3+4x-7=0$, there must be two imaginary roots.

And in the equation $x^6-1=0$ there are, for certain, four imaginary roots.

423 If an even number of consecutive terms be wanting in $f(x)$, there is at least the same number of imaginary roots.

Thus the equation $x^5+1=0$ has four terms absent; and therefore four imaginary roots at least.

THE DERIVED FUNCTIONS OF $f(x)$.

Rule for forming the derived functions.

424 *Multiply each term by the index of x, and reduce the index by one; that is, differentiate the function with respect to x.*

EXAMPLE.—Take

$$\begin{aligned}
f(x) &= x^5+x^4+x^3-x^2-x-1 \\
f^1(x) &= 5x^4+4x^3+3x^2-2x-1 \\
f^2(x) &= 20x^3+12x^2+6x-2 \\
f^3(x) &= 60x^2+24x+6 \\
f^4(x) &= 120x+24 \\
f^5(x) &= 120
\end{aligned}$$

$f^1(x)$, $f^2(x)$, &c. are called the first, second, &c. derived functions of $f(x)$.

425 To form the equation whose roots differ from those of $f(x)$ by a quantity a.

Put $x=y+a$ in $f(x)$, and expand each term by the Binomial Theorem, arranging the results in vertical columns in the following manner:—

$$\begin{aligned}
f(a+y) = {} & (a+y)^5+(a+y)^4+(a+y)^3-(a+y)^2-(a+y)-1 \\
= {} & a^5 + a^4 + a^3 - a^2 - a - 1 \\
& +(5a^4+4a^3+3a^2-2a-1)y \\
& +(10a^3+6a^2+3a-1)y^2 \\
& +(10a^2+4a+1)y^3 \\
& +(5a+1)y^4 \\
& +y^5
\end{aligned}$$

Comparing this result with that seen in (424), it is seen that

426 $$f(a+y) = f(a)+f^1(a)\,y + \frac{f^2(a)}{\underline{|2}}\,y^2 + \frac{f^3(a)}{\underline{|3}}\,y^3 + \frac{f^4(a)}{\underline{|4}}\,y^4 + \frac{f^5(a)}{\underline{|5}}\,y^5$$

so that the coefficient generally of y^r in the transformed equation is $\frac{f^r(a)}{\underline{|r}}$.

---

427 To form the equation most expeditiously when a has a numerical value, *divide $f(x)$ continuously by $x-a$, and the successive remainders will furnish the coefficients.*

EXAMPLE.—To expand $f(y+2)$ when, as in (425),

$$f(x) = x^5+x^4+x^3-x^2-x-1.$$

Divide repeatedly by $x-2$, as follows:—

	1	+ 1	+ 1	− 1	− 1	− 1	
2		+ 2	+ 6	+ 14	+ 26	+ 50	
	1	+ 3	+ 7	+ 13	+ 25	+ 49 $= f(2)$	
2		+ 2	+ 10	+ 34	+ 94		
	1	+ 5	+ 17	+ 47	+ 119 $= f^1(2)$		
2		+ 2	+ 14	+ 62			
	1	+ 7	+ 31	+ 109 $= \frac{f^2(2)}{\underline{\|2}}$			
2		+ 2	+ 18				
	1	+ 9	+ 49 $= \frac{f^3(2)}{\underline{\|3}}$				
2		+ 2					
	1	+ 11 $= \frac{f^4(2)}{\underline{\|4}}$					
	1 $= \frac{f^5(2)}{\underline{\|5}}$						

That these remainders are the required coefficients is seen by inspecting the form of the equation (426); for if that equation be divided by $x-a=y$ repeatedly, these remainders are obviously produced when $a=2$.

Thus the equation, whose roots are each less by 2 than the roots of the proposed equation, is $y^5+11y^4+49y^3+109y^2+119y+49=0$.

---

428 To make any assigned term vanish in the transformed equation, a must be so determined that the coefficient of that term shall vanish.

EXAMPLE.—In order that there may be no term involving y^4 in equation (426), we must have $f^4(a)=0$.

Find $f^4(a)$ as in (424);

thus $120a+24=0$; $\therefore\ a=-\frac{1}{5}$.

The equation in (424) must now be divided repeatedly by $x+\frac{1}{5}$ after the manner of (427), and the resulting equation will be minus its second term.

429 Note, that to remove the second term of the equation $f(x)=0$, the requisite value of a is $= -\frac{p_1}{np_0}$; that is, *the coefficient of the second term, with the sign changed, divided by the coefficient of the first term, and by the number expressing the degree of the equation.*

430 To transform $f(x)$ into an equation in y so that $y=\phi(x)$, a given function of x, *put* $x=\phi^{-1}(y)$, *the inverse function of* y.

EXAMPLE.—To obtain an equation whose roots are respectively three times the roots of the equation $x^3-6x+1=0$. Here $y=3x$; therefore $x=\frac{y}{3}$, and the equation becomes $\frac{y^3}{27}-\frac{6y}{3}+1=0$, or $y^3-54y+27=0$.

431 To transform $f(x)=0$ into an equation in which the coefficient of the first term shall be unity, and the other coefficients the least possible integers.

EXAMPLE.—Take the equation

$$288x^3+240x^2-176x-21=0.$$

Divide by the coefficient of the first term, and reduce the fractions; the equation becomes $$x^3+\frac{5}{6}x^2-\frac{11}{18}x-\frac{7}{96}=0.$$

Substitute $\frac{y}{k}$ for x, and multiply by k^3; we get

$$y^3+\frac{5k}{6}y^2-\frac{11k^2}{18}y-\frac{7k^3}{96}=0.$$

Next resolve the denominators into their prime factors,

$$y^3+\frac{5k}{2.3}y^2-\frac{11k^2}{2.3^2}y-\frac{7k^3}{2^5.3}=0.$$

The smallest value must now be assigned to k, which will suffice to make each coefficient an integer. This is easily seen by inspection to be $2^2.3=12$, and the resulting equation is $y^3+10y^2-88y-126=0$, the roots of which are connected with the roots of the original equation by the relation $y=12x$.

EQUAL ROOTS OF AN EQUATION.

By expanding $f(x+z)$ in powers of z by (405), and also by (426), and equating the coefficients of z in the two expansions, it is proved that

432 $$f'(x) = \frac{f(x)}{(x-a)} + \frac{f(x)}{(x-b)} + \frac{f(x)}{(x-c)} + \&c.,$$

from which result it appears that, if the roots a, b, c, &c. are all unequal, $f(x)$ and $f'(x)$ can have no common measure involving x. If, however, there are r roots each equal to a, s roots equal to b, t roots equal to c, &c., so that

$$f(x) = p_0(x-a)^r (x-b)^s (x-c)^t \dots$$

then

433 $$f'(x) = \frac{rf(x)}{x-a} + \frac{sf(x)}{x-b} + \frac{tf(x)}{(x-b)} + \&c.;$$

and the greatest common measure of $f(x)$ and $f'(x)$ will be

444 $$(x-a)^{r-1}(x-b)^{s-1}(x-c)^{t-1}\dots$$

When $x = a$, $f(x)$, $f'(x)$, ... $f^{r-1}(x)$ all vanish. Similarly when $x = b$, &c.

Practical method of finding the equal roots.

445 Let $f(x) = X_1 X_2^2 X_3^3 X_4^4 X_5^5 \dots X_m^m$, where

$X_1 \equiv$ product of all the factors like $(x-a)$,

$X_2^2 \equiv$ " " $(x-a)^2$,

$X_3^3 \equiv$ " " $(x-a)^3$.

Find the greatest common measure of $f(x)$ and $f'(x) = F_1(x)$ say,

" " $F_1(x)$ and $F_1'(x) = F_2(x)$,

" " $F_2(x)$ and $F_2'(x) = F_3(x)$,

... ...

... ...

Lastly, the greatest common measure of $F_{m-1}(x)$ and $F'_{m-1}(x) = F_m(x) = 1$.

Next perform the divisions

$$f(x) \div F_1(x) = \phi_1(x) \text{ say},$$
$$F_1(x) \div F_2(x) = \phi_2(x),$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$F_{m-1}(x) \div 1 = \phi_m(x).$$

And, finally,

$$\phi_1(x) \div \phi_2(x) = X_1,$$
$$\phi_2(x) \div \phi_3(x) = X_2,$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$\phi_{m-1}(x) \div \phi_m(x) = X_{m-1},$$
$$F_{m-1}(x) = \phi_m(x) = X_m.$$

[T. 82.

The solution of the equations $X_1 = 0$, $X_2 = 0$, &c. will furnish all the roots of $f(x)$; those which occur twice being found from X_2; those which occur three times each, from X_3; and so on.

446 If $f(x)$ has all its coefficients commensurable, X_1, X_2, X_3, &c. have likewise their coefficients commensurable.

Hence, if only one root be repeated r times, that root must be commensurable.

447 In all the following theorems, unless otherwise stated, $f(x)$ is understood to have unity for the coefficient of its first term.

LIMITS OF THE ROOTS.

448 If the greatest negative coefficients in $f(x)$ and $f(-x)$ be p and q respectively, then $p+1$ and $-(q+1)$ are limits of the roots.

449 If x^{n-r} and x^{n-s} are the highest negative terms in $f(x)$ and $f(-x)$ respectively, $(1+\sqrt[r]{p})$ and $-(1+\sqrt[s]{q})$ are limits of the roots.

450 If k be a superior limit to the positive roots of $f\left(\frac{1}{x}\right)$, then $\frac{1}{k}$ will be an inferior limit to the positive roots of $f(x)$.

451 If each negative coefficient be divided by the sum of all the preceding positive coefficients, the greatest of the fractions so formed + unity will be a superior limit to the positive roots.

452 *Newton's method.*—Put $x=h+y$ in $f(x)$; then, by (426),

$$f(h+y) = f(h) + yf'(h) + \frac{y^2}{\underline{|2}} f^2(h) + \ldots + \frac{y^n}{\underline{|n}} f^n(h) = 0.$$

Take h so that $f(h)$, $f'(h)$, $f^2(h) \ldots f^n(h)$ are all positive; then h is a superior limit to the positive roots.

453 According as $f(a)$ and $f(b)$ have the same or different signs, the number of roots intermediate between a and b is even or odd.

454 *Rolle's Theorem.*—One real root of the equation $f'(x)$ lies between every two adjacent real roots of $f(x)$.

455 Cor. 1.—$f(x)$ cannot have more than one root greater than the greatest root in $f'(x)$; or more than one less than the least root in $f'(x)$.

456 Cor. 2.—If $f(x)$ has m real roots, $f^r(x)$ has at least $m-r$ real roots.

457 Cor. 3.—If $f^r(x)$ has μ imaginary roots, $f(x)$ has also μ at least.

458 Cor. 4.—If $\alpha, \beta, \gamma \ldots \kappa$ be the roots of $f'(x)$; then the number of changes of sign in the series of terms

$$f(\infty),\ f(\alpha),\ f(\beta),\ f(\gamma) \ldots f(-\infty)$$

is equal to the number of roots of $f(x)$.

NEWTON'S METHOD OF DIVISORS.

459 To discover the integral roots of an equation.

Example.—To ascertain if 5 be a root of

$$x^4 - 6x^3 + 86x^2 - 176x + 105 = 0.$$

If 5 be a root it will divide 105. Add the quotient to the next coefficient. Result, -155.

If 5 be a root it will divide -155. Add the quotient to the next coefficient; and so on.

If the number tried be a root, the divisions will be effectible to the end, and the last quotient will be -1, or $-p_0$, if p_0 be not unity.

$$\begin{array}{r} 5\,)\,\underline{105} \\ 21 \\ \underline{-176} \\ 5\,)\,\underline{-155} \\ -31 \\ \underline{86} \\ 5\,)\,\underline{55} \\ 11 \\ \underline{-6} \\ 5\,)\,\underline{-5} \\ -1 \end{array}$$

460 In employing this method, limits of the roots may first be found, and divisors chosen between those limits.

461 Also, to lessen the number of trial divisors, take any integer m; then any divisor a of the last term can be rejected if $a-m$ does not divide $f(m)$.

In practice take $m = +1$ and -1.

To find whether any of the roots determined as above are repeated, divide $f(x)$ by the factors corresponding to them, and then apply the method of divisors to the resulting equation.

EXAMPLE.—Take the equation

$$x^6+2x^5-17x^4-26x^3+88x^2+72x-144=0.$$

Putting $x=1$, we find $f(1)=-24$. The divisors of 144 are

$$1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 9,\ 12,\ 16,\ 24,\ \&c.$$

The values of $a-m$ (since $m=1$) are therefore

$$0,\ 1,\ 2,\ 3,\ 5,\ 7,\ 8,\ 11,\ 15,\ 23,\ \&c.$$

Of these last numbers only 1, 2, 3, and 8 will divide 24. Hence 2, 3, 4, and 9 are the only divisors of 144 which it is of use to try. The only integral roots of the equation will be found to be ± 2 and ± 3.

462 If $f(x)$ and $F(X)$ have common roots, they are contained in the greatest common measure of $f(x)$ and $F(X)$.

463 If $f(x)$ has for its roots a, $\phi(a)$, b, $\phi(b)$ amongst others; then the equations $f(x)=0$ and $f\{\phi(x)\}=0$ have the common roots a and b.

464 But, if all the roots occur in pairs in this way, these equations coincide.

For example, suppose that each pair of roots, a and b, satisfies the equation $a+b=2r$. We may then assume $a-b=2z$. Therefore $f(z+r)=0$. This equation involves only even powers of z, and may be solved for z^2.

465 Otherwise: Let $ab=z$; then $f(x)$ is divisible by $(x-a)(x-b)$ $=x^2-2rx+z$. Perform the division until a remainder is obtained of the form $Px+Q$, where P and Q only involve z.

The equations $P=0$, $Q=0$ determine z, by (462); and a and b are found from $a+b=2r$, $ab=z$.

RECIPROCAL EQUATIONS.

466 A reciprocal equation has its roots in pairs of the form a, $\dfrac{1}{a}$; also the relation between the coefficients is

$$p_r=p_{n-r}, \text{ or else } p_r=-p_{n-r}.$$

467 A reciprocal equation *of an even degree, with its last term positive*, may be made to depend upon the solution of an equation of half the same degree.

468 EXAMPLE: $4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0$ is a reciprocal equation of an even degree, with its last term positive.

Any reciprocal equation which is not of this form may be reduced to it *by dividing by* $x+1$ *if the last term be positive; and, if the last term be negative, by dividing by* $x-1$ *or* x^2-1, *so as to bring the equation to an even degree.* Then proceed in the following manner :—

469 First bring together equidistant terms, and divide the equation by x^3; thus

$$4\left(x^3+\frac{1}{x^3}\right)-24\left(x^2+\frac{1}{x^2}\right)+57\left(x+\frac{1}{x}\right)-73=0.$$

By putting $x+\frac{1}{x}=y$, and by making repeated use of the relation $x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2$, the equation is reduced to a cubic in y, the degree being one-half that of the original equation.

Put p for $x+\frac{1}{x}$, and p_m for $x_m+\frac{1}{x_m}$.

470 The relation between the successive factors of the form p_m may be expressed by the equation

$$p_m=pp_{m-1}-p_{m-2}.$$

471 The equation for p_m, in terms of p, is

$$p_m=p^m-mp^{m-2}+\frac{m(m-3)}{1.2}p^{m-4}-\dots$$
$$+(-1)^r\frac{m(m-r-1)\dots(m-2r+1)}{\lfloor r}p^{m-2r}+\dots$$

By (545), putting $q=1$.

BINOMIAL EQUATIONS.

472 If a be a root of $x^n-1=0$, then a^m is likewise a root where m is any positive or negative integer.

473 If a be a root of $x^n+1=0$, then a^{2m+1} is likewise a root.

474 If m and n be prime to each other, x^m-1 and x^n-1 have no common root but unity.

Take $pm-qn=1$ for an indirect proof.

475 If n be a prime number, and if a be a root of $x^n-1=0$, the other roots are $a, a^2, a^3 \dots a^n$.

These are all roots, by (472). Prove, by (474), that no two can be equal.

476 If n be not a prime number, other roots besides these may exist. The successive powers, however, of some root will furnish all the rest.

477 If $x^n-1=0$ has the index $n=mpq$; m, p, q being prime factors; then the roots are the terms of the product

$$(1+\alpha+\alpha^2+\dots+\alpha^{m-1})(1+\beta+\beta^2+\dots+\beta^{p-1})$$
$$\times(1+\gamma+\gamma^2+\dots+\gamma^{q-1}),$$

where α is a root of x^m-1,
β „ x^p-1,
γ „ x^q-1,

but neither α, β, nor $\gamma=1$. Proof as in (475).

478 If $n=m^3$, and

α be a root of $x^m-1=0$,
β „ $x^m-\alpha=0$,
γ „ $x^m-\beta=0$;

then the roots of $x^n-1=0$ will be the terms of the product

$$(1+\alpha+\alpha^2+\dots+\alpha^{m-1})(1+\beta+\beta^2+\dots+\beta^{m-1})$$
$$\times(1+\gamma+\gamma^2+\dots+\gamma^{m-1}).$$

479 $x^n\pm1=0$ may be treated as a reciprocal equation, and depressed in degree after the manner of (468).

480 The complete solution of the equation

$$x^n-1=0$$

is obtained by De Moivre's Theorem. (757)

The n different roots are given by the formula

$$x=\cos\frac{2r\pi}{n}\pm\sqrt{-1}\,\sin\frac{2r\pi}{n}$$

in which r must have the successive values 0, 1, 2, 3, &c., concluding with $\frac{n}{2}$, if n be even; and with $\frac{n-1}{2}$, if n be odd.

481 Similarly the n roots of the equation

$$x^n + 1 = 0$$

are given by the formula

$$x = \cos\frac{(2r+1)\pi}{n} \pm \sqrt{-1}\,\sin\frac{(2r+1)\pi}{n}$$

r taking the successive values 0, 1, 2, 3, &c., up to $\frac{n-2}{2}$, if n be even; and up to $\frac{n-3}{2}$, if n be odd.

482 The number of different values of the product

$$A^{\frac{1}{m}}\, B^{\frac{1}{m}}$$

is equal to the least common multiple of m and n, when m and n are integers.

CUBIC EQUATIONS.

483 To solve the general cubic equation

$$x^3+px^2+qx+r=0.$$

Remove the term px^2 by the method of (429). Let the transformed equation be $x^3+qx+r=0$.

484 *Cardan's method.*—The complete theoretical solution of this equation by Cardan's method is as follows:—

Put $$x = y+z \qquad \text{(i.)}$$

$$y^3+z^3+(3yz+q)(y+z)+r=0.$$

Put $$3yz+q=0; \quad \therefore \quad y = -\frac{q}{3z}.$$

Substitute this value of y, and solve the resulting quadratic in y^3. The roots are equal to y^3 and z^3 respectively; and we have, by (i.),

485 $$x = \left\{-\frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right\}^{\frac{1}{3}} + \left\{-\frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right\}^{\frac{1}{3}}.$$

The cubic must have one real root at least, by (409).

Let m be one of the three values of $\left\{-\frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right\}^{\frac{1}{3}}$, and n one of the three values of $\left\{-\frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right\}^{\frac{1}{3}}$.

486 Let 1, a, a^2 be the three cube roots of unity, so that

$$a=-\frac{1}{2}+\frac{1}{2}\sqrt{-3}, \text{ and } a^2=-\frac{1}{2}-\frac{1}{2}\sqrt{-3}. \qquad [472$$

487 Then, since $\sqrt[3]{m^3}=m\sqrt[3]{1}$, the roots of the cubic will be

$$m+n, \quad am+a^2n, \quad a^2m+an.$$

Now, if in the expansion of

$$\left\{-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right\}^{\frac{1}{3}}$$

by the Binomial Theorem, we put

μ = the sum of the odd terms, and

ν = the sum of the even terms;

then we shall have $m=\mu+\nu$, and $n=\mu-\nu$;

or else $m=\mu+\nu\sqrt{-1}$, and $n=\mu-\nu\sqrt{-1}$;

according as $\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}$ is real or imaginary.

By substituting these expressions for m and n in (487), it appears that—

488 (i.) If $\frac{r^2}{4}+\frac{q^3}{27}$ be positive, the roots of the cubic will be

$$2\mu, \quad -\mu+\nu\sqrt{-3}, \quad -\mu-\nu\sqrt{-3}.$$

(ii.) If $\frac{r^2}{4}+\frac{q^3}{27}$ be negative, the roots will be

$$2\mu, \quad -\mu+\nu\sqrt{3}, \quad -\mu-\nu\sqrt{3}.$$

(iii.) If $\frac{r^2}{4}+\frac{q^3}{27}=0$, the roots are

$$2m, \quad -m, \quad -m;$$

since m is now equal to μ.

489 *The Trigonometrical method.*—The equation

$$x^3+qx+r=0$$

may be solved in the following manner, by Trigonometry, when $\frac{r^2}{4}+\frac{q^3}{27}$ is negative.

Assume $x=n\cos a$. Divide the equation by n^3; thus

$$\cos^3 a+\frac{q}{n^2}\cos a+\frac{r}{n^3}=0.$$

But $$\cos^3 a-\frac{3}{4}\cos a-\frac{\cos 3a}{4}=0. \qquad \text{By (657)}$$

Equate coefficients in the two equations; the result is

$$n = \left(\frac{4q}{3}\right)^{\frac{1}{2}}, \qquad \cos 3a = -4r\left(-\frac{3}{4q}\right)^{\frac{3}{2}},$$

a must now be found with the aid of the Trigonometrical tables.

490 The roots of the cubic will be

$$n\cos a, \quad n\cos\left(\tfrac{2}{3}\pi+a\right), \quad n\cos\left(\tfrac{2}{3}\pi-a\right).$$

491 Observe that, according as $\frac{r^2}{4}+\frac{q^3}{27}$ is positive or negative, Cardan's method or the Trigonometrical will be practicable. In the former case, there will be *one real and two imaginary roots;* in the latter case, *three real roots.*

BIQUADRATIC EQUATIONS.

492 *Descartes' Solution.*—To solve the equation

$$x^4+qx^2+rx+s=0 \quad \text{(i.)}$$

the term in x^3 having been removed by the method of (429).

Assume $\quad (x^2+ex+f)(x^2-ex+g)=0 \quad$ (ii.)

Multiply out, and equate coefficients with (i.); and the following equations for determining f, g, and e are obtained

$$g+f=q+e^2, \quad g-f=\frac{r}{e}, \quad gf=s \quad \text{(iii.)}$$

493 $$e^6+2qe^4+(q^2-4s)e^2-r^2=0 \quad \text{(iv.)}$$

494 *The cubic in e^2 is reducible by Cardan's method, when the biquadratic has two real and two imaginary roots.* For proof, take $a\pm i\beta$ and $-a\pm\gamma$ as the roots of (i.), since their sum must be zero. Form the sum of each pair for the values of e [see (ii.)], and apply the rules in (488) to the cubic in e^2.

If the biquadratic has all its roots real, or all imaginary, the cubic will have all its roots real. Take $a\pm i\beta$ and $-a\pm i\gamma$ for four imaginary roots of (i.), and form the values of e as before.

495 *If a^2, β^2, γ^2 be the roots of the cubic in e^2, the roots of the biquadratic will be* $-\frac{1}{2}(a+\beta+\gamma)$, $\frac{1}{2}(a+\beta-\gamma)$, $\frac{1}{2}(\beta+\gamma-a)$, $\frac{1}{2}(\gamma+a-\beta)$.

For proof, take w, x, y, z for the roots of the biquadratic; then, by (ii.), the sum of each pair must give a value of e. Hence, we have only to solve the symmetrical equations

$$\begin{aligned} y+z &= \alpha, & w+x &= -\alpha, \\ z+x &= \beta, & w+y &= -\beta, \\ x+y &= \gamma, & w+z &= -\gamma. \end{aligned}$$

496 *Ferrari's solution.*—To the left member of the equation

$$x^4+px^3+qx^2+rx+s=0,$$

add the quantity $ax^2+bx+\frac{b^2}{4a}$, and assume the result

$$= \left(x^2+\frac{p}{2}x+m\right)^2.$$

497 Expanding and equating coefficients, the following cubic equation for determining m is obtained

$$8m^3-4qm^2+(2pr-8s)m+4qs-p^2s-r=0.$$

Then x is given by the two quadratics

$$x^2+\frac{p}{2}x+m=\pm\frac{2ax+b}{2\sqrt{a}}$$

498 *The cubic in m is reducible by Cardan's method when the biquadratic has two real and two imaginary roots.* Assume α, β, γ, δ for the roots of the biquadratic; then $\alpha\beta$ and $\gamma\delta$ are the respective products of roots of the two quadratics above. From this find m in terms of $\alpha\beta\gamma\delta$.

499 *Euler's solution.*—Remove the term in x^3; then we have

$$x^4+qx^2+rx+s=0.$$

500 Assume $x=y+z+u$, and it may be shewn that y^2, z^2, and u^2 are the roots of the equation

$$t^3+\frac{q}{2}t^2+\frac{q^2-4s}{16}t-\frac{r^2}{64}=0.$$

501 The six values of y, z, and u, thence obtained, are restricted by the relation $yzu=-\frac{r}{8}$.

Thus $x=y+z+u$ will take four different values.

COMMENSURABLE ROOTS.

502 To find the commensurable roots of an equation.

First transform it by putting $x = \frac{y}{k}$ into an equation of the form $$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0,$$ having $p_0 = 1$, and the remaining coefficients integers. (431)

503 This equation cannot have a rational fractional root, and the integral roots may be found by Newton's method of Divisors (459).

These roots, divided each by k, will furnish the commensurable roots of the original equation.

504 Example.—To find the commensurable roots of the equation
$$81x^5 - 207x^4 - 9x^3 + 89x^2 + 2x - 8 = 0.$$

Dividing by 81, and proceeding as in (431), we find the requisite substitution to be $$x = \frac{y}{9}.$$
The transformed equation is
$$y^5 - 23y^4 - 9y^3 + 801y^2 + 162y - 5832 = 0.$$
The roots all lie between 24 and -34, by (451).

The method of divisors gives the integral roots
$$6,\ -4,\ \text{and } 3.$$
Therefore, dividing each by 9, we find the commensurable roots of the original equation to be $$\tfrac{2}{3},\ -\tfrac{4}{9},\ \text{and } \tfrac{1}{3}.$$

505 To obtain the remaining roots; diminish the transformed equation by the roots 6, -4, and 3, in the following manner (see 427):—

	1−23− 9+801+162−5832
6	6−102−666+810+5832
	1−17−111+135+972
−4	− 4+ 84+108−972
	1−21− 27+243
3	3− 54−243
	1−18− 81

The depressed equation is therefore
$$y^2 - 18y - 81 = 0.$$
The roots of which are $9(1+\sqrt{2})$ and $9(1-\sqrt{2})$; and, consequently, the incommensurable roots of the proposed equation are $1+\sqrt{2}$ and $1-\sqrt{2}$.

INCOMMENSURABLE ROOTS.

506 *Sturm's Theorem.*—If $f(x)$, freed from equal roots, be divided by $f'(x)$, and the last divisor by the last remainder, *changing the sign* of each remainder before dividing by it, until a remainder independent of x is obtained, or else a remainder which cannot change its sign; then $f(x)$, $f'(x)$, and the successive remainders constitute Sturm's functions, and are denoted by $f(x), f_1(x), f_2(x), \&c. \ldots\ldots f_m(x)$.

The operation may be exhibited as follows:—

$$f(x) = q_1 f_1(x) - f_2(x),$$
$$f_1(x) = q_2 f_2(x) - f_3(x),$$
$$f_2(x) = q_3 f_3(x) - f_4(x),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$f_{m-2}(x) = q_{m-1} f_{m-1}(x) - f_m(x).$$

507 NOTE.—Any constant factor of a remainder may be rejected, and the quotient may be set down for the corresponding function.

---

508 An inspection of the foregoing equations shews—

(1) That $f_m(x)$ cannot be zero; for, if it were, $f(x)$ and $f_1(x)$ would have a common factor, and therefore $f(x)$ would have equal roots, by (432).

(2) Two consecutive functions, after the first, cannot vanish together; for this would make $f_m(x)$ zero.

(3) When any function, after the first, vanishes, the two adjacent ones have contrary signs.

509 *If, as x increases, $f(x)$ passes through the value zero, Sturm's functions lose one change of sign.*

For, before $f(x)$ takes the value zero, $f(x)$ and $f_1(x)$ have contrary signs, and afterwards they have the same sign; as may be shewn by making h small, and changing its sign in the expansion of $f(x+h)$, by (426).

510 *If any other of Sturm's functions vanishes, there is neither loss nor gain in the number of changes of sign.*

This will appear on inspecting the equations.

511 RESULT.—*The number of roots of $f(x)$ between a and b is equal to the difference in the number of changes of sign in Sturm's functions, when $x=a$ and when $x=b$.*

512 Cor.—The total number of roots of $f(x)$ will be found by taking $a = +\infty$ and $b = -\infty$; the sign of each function will then be the same as that of its first term.

When the number of functions exceeds the degree of $f(x)$ by unity, the two following theorems hold :—

513 *If the first terms in all the functions, after the first, are positive; all the roots of $f(x)$ are real.*

514 *If the first terms are not all positive; then, for every change of sign, there will be a pair of imaginary roots.*

For the proof put $x = +\infty$ and $-\infty$, and examine the number of changes of sign in each case, applying Descartes' rule. (416).

515 If $\phi(x)$ has no factor in common with $f(x)$, and if $\phi(x)$ and $f'(x)$ take the same sign when $f(x) = 0$; then the rest of Sturm's functions may be found from $f(x)$ and $\phi(x)$, instead of $f'(x)$. For the reasoning in (509) and (510) will apply to the new functions.

516 If Sturm's functions be formed without first removing equal roots from $f(x)$, the theorem will still give the number of distinct roots, without repetitions, between assigned limits.

For if $f(x)$ and $f_1(x)$ be divided by their highest common factor (see 444), and if the quotients be used instead of $f(x)$ and $f_1(x)$ to form Sturm's functions; then, by (515), the theorem will apply to the new set of functions, which will differ only from those formed from $f(x)$ and $f_1(x)$ by the absence of the same factor in every term of the series.

517 Example.—To find the position of the roots of the equation

$$x^4-4x^3+x^2+6x+2 = 0.$$

Sturm's functions, formed according to the rule given above, are here calculated.

The first terms of the functions are all positive; therefore there is no imaginary root.

$$\begin{aligned} f(x) &= x^4-4a^3+x^2+6x+2 \\ f_1(x) &= 2x^3-6x^2+x+3 \\ f_2(x) &= 5x^2-10x-7 \\ f_3(x) &= x-1 \\ f_4(x) &= 12 \end{aligned}$$

The changes of sign in the functions, as x passes through integral values, are exhibited in the adjoining table. There are two changes of sign lost while x passes from -1 to 0, and two more lost while x passes from 2 to 3. There

x =	−2	−1	0	1	2	3	4
$f(x)$ =	+	+	+	+	+	+	+
$f_1(x)$ =	−	−	+	+	−	+	+
$f_2(x)$ =	+	+	−	−	−	+	+
$f_3(x)$ =	−	−	−	+	+	+	+
$f_4(x)$ =	+	+	+	+	+	+	+
No. of changes of sign ……	4	4	2	2	2	0	0

are therefore two roots lying between 0 and -1; and two roots also between 2 and 3.

These roots are all incommensurable, by (503).

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518 ***Fourier's Theorem.***—Fourier's functions are the following quantities $f(x),\ f'(x),\ f''(x) \ldots\ldots f^n(x)$.

519 Properties of Fourier's functions.—As x increases, Fourier's functions lose *one* change of sign for each root of the equation $f(x) = 0$, through which x passes, and r changes of sign for r repeated roots.

520 If any of the other functions vanish, an *even* number of changes of sign is lost.

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521 Results.—*The number of real roots of $f(x)$ between α and β cannot be more than the difference between the number of changes of sign in Fourier's functions when $x = \alpha$, and the number of changes when $x = \beta$.*

522 When that difference is *odd*, the number of intermediate roots is *odd*, and therefore *one at least*.

523 When the same difference is *even*, the number of intermediate roots is either *even or zero*.

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524 Descartes' rule of signs follows from the above for the signs of Fourier's functions, when $x = 0$ are the signs of the terms in $f(x)$; and when $x = \infty$, Fourier's functions are all positive.

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525 ***Lagrange's method of approximating to the incommensurable roots of an equation.***

Let a be the greatest integer less than an incommensurable root of $f(x)$. Diminish the roots of $f(x)$ by a. Take the reciprocal of the resulting equation. Let b be the greatest integer less than a positive root of this equation. Diminish the roots of this equation by b, and proceed as before.

526 Let a, b, c, &c. be the quantities thus determined; then, an approximation to the incommensurable root of $f(x)$ will be the continued fraction $x = a + \frac{1}{b+}\,\frac{1}{c+}$

527 *Newton's method of approximation.*—If c_1 be a quantity a little less than one of the roots of the equation $f(x) = 0$, so that $f(c_1+h) = 0$; then c_1 is a first approximation to the value of the root. Also because

$$f(c_1+h) = f(c_1)+hf'(c_1)+\frac{h^2}{\underline{|2}}f''(c_1)+\&c. \ldots\ldots (426),$$

and h is but small, a second approximation to the root will be

$$c_1 - \frac{f(c_1)}{f'(c_1)} = c_2.$$

In the same way a third approximation may be obtained from c_2, and so on.

528 *Fourier's limitation of Newton's method.*—To ensure that c_1, c_2, c_3, &c. shall successively increase up to the value c_1+h without passing beyond it, it is necessary for all values of x between c_1 and c_1+h.

(i.) *That $f(x)$ and $f'(x)$ should have contrary signs.*

(ii.) *That $f(x)$ and $f''(x)$ should have the same sign.*

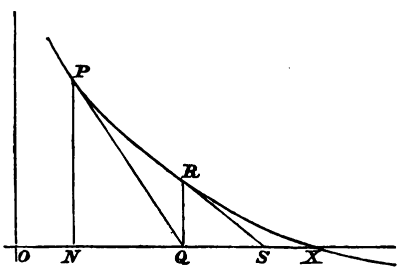

FIG. 1.

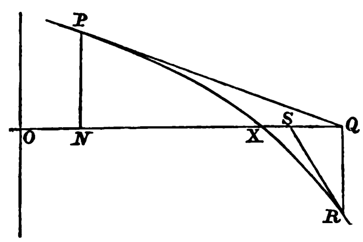

FIG. 2.

A proof may be obtained from the figure. Draw the curve $y = f(x)$. Let OX be a root of the equation, and $ON = c_1$; draw the successive ordinates and tangents NP, PQ, QR, &c. Then $OQ = c_2$, $OS = c_3$, and so on.

Fig. (2) represents $c_2 > OX$, and the subsequent approximations decreasing towards the root.

530 *Newton's Rule for Limits of the Roots.*—Let the coefficients of $f(x)$ be respectively divided by the Binomial coefficients, and let a_0, a_1, a_2 ... a_n be the quotients, so that

$$f(x) = a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{1.2}a_2x^{n-2} + \ldots + na_{n-1}x + a_n.$$

Let $A_1, A_2, A_3 \dots A_n$ be formed by the law $A_r = a_r^2 - a_{r-1}a_{r+1}$. Write the first series of quantities over the second, in the following manner :—

$$a_0, \quad a_1, \quad a_2, \quad a_3 \dots\dots a_{n-1}, \quad a_n,$$
$$A_0, \quad A_1, \quad A_2, \quad A_3 \dots\dots A_{n-1}, \quad A_n.$$

Whenever two adjacent terms in the first series have the same sign, and the two corresponding terms below them in the second series also the same sign; let this be called *a double permanence*. When two adjacent terms above have different signs, and the two below the same sign, let this be known as *a variation-permanence.*

531 RULE.—*The number of double permanences in the associated series is a superior limit to the number of negative roots of $f(x)$.*

The number of variation-permanences is a superior limit to the number of positive roots.

The number of imaginary roots cannot be less than the number of variations of sign in the second series.

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532 ***Sylvester's Theorem.***—Let $f(x+\lambda)$ be expanded by (426) in powers of x, and let the two series be formed as in Newton's Rule (530).

Let $P(\lambda)$ denote the number of double permanences.

Then $P(\lambda) \sim P(\mu)$ is either equal to the number of roots of $f(x)$, or surpasses that number by an even integer.

NOTE.—The first series may be multiplied by $\underline{|n}$, and will then stand thus,

$$f^n(\lambda), \quad f^{n-1}(\lambda), \quad \underline{|2}\, f^{n-2}(\lambda), \quad \underline{|3}\, f^{n-3}(\lambda) \dots \underline{|n}\, f(\lambda).$$

The second series may be reduced to

$$G_n(\lambda), \quad G_{n-1}(\lambda), \quad G_{n-2}(\lambda) \dots G(\lambda),$$

where $$G_r(\lambda) \equiv \{f^r(\lambda)\}^2 - \frac{n-r+1}{n-r} f^{r-1}(\lambda) f^{r+1}(\lambda).$$

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533 ***Horner's Method.***—To find the numerical values of the roots of an equation. Take, for example, the equation

$$x^4 - 4x^3 + x^2 + 6x + 2 = 0,$$

and find limits of the roots by Sturm's Method or otherwise.

It has been shewn in (517) that this equation has two incommensurable roots between 2 and 3. The process of calculating the least of these roots is here exhibited.

	−4		+1		+6		+2 (2·414213
	2		−4		−6		0
	−2		−3		0	A_1	20000
	2		0		−6		−19584
	0		−3	B_1	−6000	A_2	4160000
	2		4		1104		−2955839
	2	C_1	100		−4896	A_3	12041610000
	2		176		1872		−11437245184
D_1	40		276	B_2	−3024000	A_4	604364816
	4		192		68161		−566003348
	44		468		−2955839	A_5	38361468
	4		208		68723		−28285470
	48	C_2	67600	B_3	−2887116000	A_6	10075998
	4		561		27804704		− 8485368
	52		68161		−2859311296	A_7	1590630
	4		562		27895072		
D_2	560		68723	B_4	−2831416224		
	1		563		139948		282843) 1590630 (562372
	561	C_3	6928600		−283001674		1414215
	1		22576		139970		28284) 176415
	562		6951176	B_5	−282861704		169706
	1		22592		700		2828) 6709
	563		6973768		−28285470		5657
	1		22608		700		282) 1052
D_3	5640	C_4	6996376	B_6	−28284770		848
	4		11		21		28) 204
	5644		69974		−2828456		197
	4		11		21		2) 7
	5648		69985	B_7	−2828435		5
	4		11				2
	5652	C_5	69996				
	4	C_6	7				
D_4	5,656						Root = 2·414213562372.

Method.—Diminish the roots by 2 in the manner of (427).

The resulting coefficients are indicated by A_1, B_1, C_1, D_1.

By Newton's rule (527), $-\frac{f(c)}{f'(c)}$; that is, $-\frac{A_1}{B_1}$ is an approximation to the remaining part of the root. This gives ·3 for the next figure; ·4 will be found to be the correct one. The highest figure must be taken which will not change the sign of A.

Diminish the roots by ·4. This is accomplished most easily by affixing ciphers to A_1, B_1, C_1, D_1, in the manner shewn, and then employing 4 instead of ·4.

Having obtained A_2, and observing that its sign is +, retrace the steps, trying 5 instead of 4. This gives A_2 with a minus sign, thereby proving the existence of a root between 2·4 and 2·5. The new coefficients are A_2, B_2, C_2, D_2.

$-\frac{A_2}{B_2}$ gives 1 for the next figure of the root.

Affix ciphers as before, and diminish the roots by 1, distinguishing the new coefficients as A_3, B_3, C_3, D_3.

Note that at every stage of the work A and B must preserve their signs unchanged. If a change of sign takes place it shews that too large a figure has been tried.

To abridge the calculation proceed thus:—After a certain number of figures of the root have been obtained (in this example four), instead of adding ciphers cut off one digit from B_4, two from C_4, and three from D_4. This amounts to the same thing as adding the ciphers, and then dividing each number by 10000.

Continue the work with the numbers so reduced, and cut off digits in like manner at each stage until the D and C columns have disappeared.

A_7 and B_7 now alone remain, and six additional figures of the root are determined correctly by the division of A_7 by B_7.

To find the other root which lies between 2 and 3, we proceed as follows:—After diminishing the roots by 2, try 6 for the next figure. This gives A_1 negative; 7 does the same, but 8 makes A_1 positive. That is to say, $f(2{\cdot}7)$ is negative, and $f(2{\cdot}8)$ positive. Therefore a root exists between 2·7 and 2·8, and its value may be approximated to, in the manner shewn.

Throughout this last calculation A will preserve the negative sign. Observe also that the trial number for the next figure of the root given at each stage of the process by the formula $-\frac{f(c)}{f'(c)}$, will in this case be always too great, as in the former case it was always too small.

SYMMETRICAL FUNCTIONS OF THE ROOTS OF AN EQUATION.

NOTATION.—Let a, b, c ... be the roots of the equation $f(x) = 0$.

Let s_m denote $a^m + b^m + \ldots$, the sum of the m^{th} powers of the roots.

Let $s_{m,p}$ denote $a^m b^p + b^m a^p + a^m c^p + \ldots$ through all the permutations of the roots, two at a time.

Similarly let $s_{m,p,q}$ denote $a^m b^p c^q + a^m b^p d^q + \ldots$, taking all the permutations of the roots three at a time; and so on.

534 *SUMS OF THE POWERS OF THE ROOTS.*

$$s_m+p_1s_{m-1}+p_2s_{m-2}+\ldots+p_{m-1}s_1+mp_m = 0,$$

where m is less than n, the degree of $f(x)$.

Obtained by expanding by division each term in the value of $f'(x)$ given at (432), arranging the whole in powers of x, and equating coefficients in the result and in the value of $f'(x)$, found by differentiation as in (424).

535 If m be greater than n, the formula will be

$$s_m+p_1 s_{m-1}+p_2 s_{m-2}+ \ldots +p_n s_{m-n} = 0.$$

Obtained by multiplying $f(x)=0$ by x^{m-n}, substituting for x the roots a, b, c, &c. in succession, and adding the results.

By these formulæ s_1, s_2, s_3, &c. may be calculated successively.

536 To find the sum of the negative powers of the roots, put m equal to $n-1$, $n-2$, $n-3$, &c. successively in (535), in order to obtain s_{-1}, s_{-2}, s_{-3}, &c.

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537 To calculate s_r independently.

Rule: $s_r = -r\times$ *coefficient of* x^{-r} *in the expansion of* $\log\frac{f(x)}{x^n}$ *in descending powers of* x.

Proved by taking $f(x)=(x-a)(x-b)(x-c)\ldots$, dividing by x^n, and expanding the logarithm of the right side of the equation by (156).

---

538 *SYMMETRICAL FUNCTIONS WHICH ARE NOT POWERS OF THE ROOTS.*

These are expressed in terms of the sums of powers of the roots as under, and thence, by (534), in terms of the roots explicitly,

$$s_{m,\,p} = s_m s_p - s_{m+p}\,,$$

539 $$s_{m,\,p,\,q} = s_m s_p s_q - s_{m+p}s_q - s_{m+q}s_p - s_{p+q}s_m + 2s_{m+p+q}\,.$$

The last equation may be proved by multiplying $s_{m,\,p}$ by s_q; and expansions of other symmetrical functions may be obtained in a similar way.

---

540 If $\phi(x)$ be a rational integral function of x, then the symmetrical function of the roots of $f(x)$, denoted by

$\phi(a)+\phi(b)+\phi(c)+$&c., is equal to the coefficient of x^{n-1} in the remainder obtained by dividing $\phi(x)f'(x)$ by $f(x)$.

Proved by multiplying the equation (432) by $\frac{\phi(x)}{f(x)}$, and by theorem (401).

---

541 To find the equation whose roots are the squares of the differences of the roots of a given equation.

Let $F(x)$ be the given equation, and S_r the sum of the r^{th} powers of its roots. Let $f(x)$ and s_r have the same meaning with regard to the required equation.

The coefficients of the required equation can be calculated from those of the given one as follows:—

The coefficients of each equation may be connected with the sums of the powers of its roots by (534); *and the sums of the powers of the roots of the two equations are connected by the formula*

542 $$2s_r = nS_{2r} - 2rS_1S_{2r-1} + \frac{2r(2r-1)}{1 \cdot 2}S_2S_{2r-2} - \ldots + nS_{2r}.$$

RULE.—$2s_r$ *is equal to the formal expansion of* $(S-S)^{2r}$ *by the Binomial Theorem, with the first and last terms each multiplied by* n, *and the indices all changed to suffixes.* As the equi-distant terms are equal we can divide by 2, and take half the series.

DEMONSTRATION.—Let $a, b, c \ldots$ be the roots of $F(x)$.

Let $$\phi(x) = (x-a)^{2r} + (x-b)^{2r} + \ldots \quad \text{(i.)}$$

Expand each term on the right by the Bin. Theor., and add, substituting S_1, S_2, &c. In the result change x into $a, b, c \ldots$ successively, and add the n equations to obtain the formula, observing that, by (i.),

$$\phi(a) + \phi(b) + \ldots = 2s_r.$$

If n be the degree of $F(x)$, then $\frac{1}{2}n(n-1)$ is the degree of $f(x)$. By (96).

543 The last term of the equation $f(x) = 0$ is equal to

$$n^n F(\alpha)\, F(\beta)\, F(\gamma) \ldots$$

where $\alpha, \beta, \gamma, \ldots$ are the roots of $F'(x)$. Proved by shewing that $$F'(a)\, F'(b) \ldots = n^n F(\alpha)\, F(\beta) \ldots$$

544 If $F(x)$ has negative or imaginary roots, $f(x)$ must have imaginary roots.

545 The sum of the m^{th} powers of the roots of the quadratic equation $$x^2-px+q=0.$$

$$s_m = p^m - mp^{m-2}q + \frac{m(m-3)}{\lfloor 2}p^{m-4}q^2 - \dots$$

$$\dots + (-1)^r \frac{m(m-r-1)\dots(m-2r+1)}{\lfloor r} p^{m-2r}q^r + \&c.$$

By (537) expanding the logarithm by (156).

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546 The sum of the m^{th} powers of the roots of $x^n-1=0$ is n if m be a multiple of n, and zero if it be not.

By (537); expanding the logarithm by (156).

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547 If $$\phi(x) = a_0 + a_1x + a_2x^2 + \&c. \quad \dots\dots\dots\dots \text{(i.)},$$ then the sum of the selected terms

$$a_mx^m + a_{m+n}x^{m+n} + a_{m+2n}x^{m+2n} + \&c.$$

will be $s = \dfrac{1}{n}\{\alpha^{n-m}\phi(\alpha x) + \beta^{n-m}\phi(\beta x) + \gamma^{n-m}\phi(\gamma x) + \&c.\}$

where α, β, γ, &c. are the n^{th} roots of unity.

For proof, multiply (i.) by α^{n-m}, and change x into αx; so with β, γ, &c., and add the resulting equations.

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548 To approximate to the root of an equation by means of the sums of the powers of the roots.

By taking m large enough, the fraction $\frac{s_{m+1}}{s_m}$ will approximate to the value of the numerically greatest root, unless there be a modulus of imaginary roots greater than any real root, in which case the fraction has no limiting value.

549 Similarly the fraction $\frac{s_m s_{m+2} - s_{m+1}^2}{s_{m-1}s_{m+1} - s_m^2}$ approximates, as m increases, to the *greatest product* of any pair of roots, real or imaginary; excepting in the case in which the product of the pair of imaginary roots, though less than the product of the two real roots, is greater than the square of the least of them, for then the fraction has no limiting value.

550 Similarly the fraction $\frac{s_m s_{m+3} - s_{m+1} s_{m+2}}{s_m s_{m+2} - s_{m+1}^2}$ approximates, as m increases, to the sum of the two numerically greatest roots, or to the sum of the two imaginary roots with the greatest modulus.

EXPANSION OF AN IMPLICIT FUNCTION OF x.

Let $y^\alpha(Ax^a+)+y^\beta(B_1x^b+)+\ldots+y^\sigma(Sx^s+)=0$ ......(1)

be an equation arranged in descending powers of y, the coefficients being functions of x, the highest powers only of x in each coefficient being written.

It is required to obtain y in a series of descending powers of x.

First form the fractions

$$-\frac{a-b}{\alpha-\beta},\ -\frac{a-c}{\alpha-\gamma},\ -\frac{a-d}{\alpha-\delta}\ \ldots\ldots -\frac{a-s}{\alpha-\sigma}\ \ldots\ldots\ldots(2).$$

Let $-\frac{a-k}{\alpha-n}=t$ be the greatest of these algebraically, or if several are equal and greater than the rest, let it be the last of such. Then, with the letters corresponding to these equal and greatest fractions, form the equation

$$Au^\alpha+\ldots\ldots+Ku^\kappa=0\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Each value of u in this equation corresponds to a value of y, commencing with ux^t.

Next select the greatest of the fractions

$$-\frac{k-l}{\kappa-\lambda},\ -\frac{k-m}{\kappa-\mu}\ \ldots\ldots -\frac{k-s}{\kappa-\sigma}\ \ldots\ldots\ldots\ldots\ldots(4).$$

Let $-\frac{k-n}{\kappa-\nu}=t'$ be the last of the greatest ones. Form the corresponding equation $Ku^\kappa+\ldots+Nu^\nu=0$ ....(5).

Then each value of u in this equation gives a corresponding value of y, commencing with $ux^{t'}$.

Proceed in this way until the last fraction of the series (2) is reached.

To obtain the second term in the expansion of y, put

$$y = x^t(u+y_1) \text{ in } (1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

employing the different values of u, and again of t' and u, t'' and u, &c. in succession; and in each case this substitution will produce an equation in y and x similar to the original equation in y.

Repeat the foregoing process with the new equation in y, observing the following additional rule :—

When all the values of t, t', t'', &c. have been obtained, the negative ones only must be employed in forming the equations in u. (7).

552 To obtain y in a series of *ascending* powers of x.

Arrange equation (1) so that α, β, γ, &c. may be in *ascending* order of magnitude, and a, b, c, &c. the *lowest* powers of x in the respective coefficients.

Select t, the greatest of the fractions in (2), and proceed exactly as before, with the one exception of substituting the word *positive* for *negative* in (7).

553 EXAMPLE.—Take the equation

$$(x^3+x^4)+(3x^2-5x^3)\,y+(-4x+7x^2+x^3)\,y^2-y^5=0.$$

It is required to expand y in ascending powers of x.

The fractions (2) are $-\frac{3-2}{0-1}$, $-\frac{3-1}{0-2}$, $-\frac{3-0}{0-5}$; or 1, 1, and $\frac{3}{5}$.

The first two being equal and greatest, we have $t=1$.

The fractions (4) reduce to $-\frac{1-0}{2-5}=\frac{1}{3}=t'$.

Equation (3) is $\qquad 1+3u-4u^2=0,$

which gives $\qquad u=1$ and $-\frac{1}{4}$, with $t=1$.

Equation (5) is $\qquad -4u^2-u^5=0,$

and from this $\qquad u=0$ and $-4^{\frac{1}{3}}$, with $t'=\frac{1}{3}$.

We have now to substitute for y, according to (6), either

$$x(1+y_1), \quad x(-\tfrac{1}{4}+y_1), \quad x^{\frac{1}{3}}y, \text{ or } x^{\frac{1}{3}}(-4^{\frac{1}{3}}+y_1).$$

Put $y=x(1+y_1)$, the first of these values, in the original equation, and arrange in ascending powers of y, thus

$$-4x^4+(-5x^3+)\,y_1+(-4x^3+)\,y_1^2-10x^3y_1^3-5x^3y_1^4-x^5y_1^5=0.$$

The lowest power only of x in each coefficient is here written.

The fractions (2) now become

$$-\frac{4-3}{0-1},\ -\frac{4-3}{0-2},\ -\frac{4-5}{0-3},\ -\frac{4-5}{0-4},\ -\frac{4-5}{0-5};$$

or $$1,\ \tfrac{1}{2},\ -\tfrac{1}{3},\ -\tfrac{1}{4},\ -\tfrac{1}{5}.$$

From these $t=1$, and equation (3) becomes

$$-4-5u=0;\ \therefore\ u=-\tfrac{4}{5}.$$

Hence one of the values of y_1 is, as in (6), $y_1=x(-\tfrac{4}{5}+y_2)$.

Therefore $$y=x\{1+x(-\tfrac{4}{5}+y_2)\}=x-\tfrac{4}{5}x^2+\ldots$$

Thus the first two terms of one of the expansions have been obtained.

DETERMINANTS.

554 *Definitions.*—The determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is equivalent to $a_1b_2-a_2b_1$, and is called a determinant of the second order. A determinant of the third order is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1).$$

Another notation is $\Sigma \pm a_1b_2c_3$, or simply $(a_1b_2c_3)$.

The letters are named *constituents*, and the terms are called *elements*. The determinant is composed of all the elements obtained by permutations of the suffixes 1, 2, 3.

The coefficients of the constituents are determinants of the next lower order, and are termed *minors* of the original determinant. Thus, the first determinant above is the minor of c_3 in the second determinant. It is denoted by C_3. So the minor of a_1 is denoted by A_1, and so on.

555 A determinant of the n^{th} order may be written in either of the forms below

$$\begin{vmatrix} a_1 & a_2 & \ldots & a_r & \ldots & a_n \\ b_1 & b_2 & \ldots & b_r & \ldots & b_n \\ \ldots & \ldots & & \ldots & & \ldots \\ l_1 & l_2 & \ldots & l_r & \ldots & l_n \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1r} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2r} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots & & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nr} & \ldots & a_{nn} \end{vmatrix}$$

In the latter, or double suffix notation, the first suffix indicates the row, and the second the column. The former notation will be adopted in these pages.

A *Composite determinant* is one in which the number of columns exceeds the number of rows, and it is written as in the annexed example. $\left\|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}\right\|$ Its value is the sum of all the determinants obtained by taking a number of rows in every possible way.

A *Simple determinant* has single terms for its constituents.

A *Compound determinant* has more than one term in some or all of its constituents. See (570) for an example.

For the definitions of *Symmetrical*, *Reciprocal*, *Partial*, and *Complementary* determinants; see (574), (575), and (576).

General Theory.

556 The number of constituents is n^2.

The number of elements in the complete determinant is $\lfloor n$.

557 The first or leading element is $a_1 b_2 c_3 \ldots l_n$. Any element may be derived from the first by permutation of the suffixes.

The sign of an element is $+$ or $-$ according as it has been obtained from the diagonal element by an even or odd number of permutations of the suffixes.

Hence the following rule for determining the sign of an element.

Rule.—*Take the suffixes in order, and put them back to their places in the first element. Let m be the whole number of places passed over; then $(-1)^m$ will give the sign required.*

Ex.—To find the sign of the element $a_4 b_3 c_5 d_1 e_2$ of the determinant $(a_1 b_2 c_3 d_4 e_5)$.

	a_4	b_3	c_5	d_1	e_2
Move the suffix 1, three places... ...	1	4	3	5	2
" " 2, three places... ...	1	2	4	3	5
" " 3, one place	1	2	3	4	5

In all, seven places; therefore $(-1)^7 = -1$ gives the sign required.

558 If two suffixes in any element be transposed, the sign of the element is changed.

Half of the elements are plus, and half are minus.

559 The elements are not altered by changing the rows into columns.

If two rows or columns are transposed, the sign of the determinant is changed. Because each element changes its sign.

If two rows or columns are identical, the determinant vanishes.

560 If all the constituents but one in a row or column vanish, the determinant becomes the product of that constituent and a determinant of the next lower order.

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561 A cyclical interchange is effected by $n-1$ successive transpositions of adjacent rows or columns, until the top row has been brought to the bottom, or the left column to the right side. Hence

A cyclical interchange changes the sign of a determinant of an even order only.

The r^{th} row may be brought to the top by $r-1$ cyclical interchanges.

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562 If each constituent in a row or column be multiplied by the same factor, the determinant becomes multiplied by it.

If each constituent of a row or column is the sum of m terms, the compound determinant becomes the sum of m simple determinants of the same order.

Also, if every constituent of the determinant consists of m terms, the compound determinant is resolvable into the sum of m^2 simple determinants.

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563 To express the minor of the r^{th} row and k^{th} column as a determinant of the $n-1^{\text{th}}$ order.

Put all the constituents in the r^{th} row and k^{th} column equal to 0, and then make $r-1$ cyclical interchanges in the rows and $k-1$ in the columns, and multiply by $(-1)^{(r+k)(n-1)}$.

[$\because\ = (-1)^{(r-1+k-1)(n-1)}$.

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564 To express a determinant as a determinant of a higher order.

Continue the diagonal with constituents of "ones," and fill up with zeros on one side, and with any quantities whatever (α, β, γ, &c.) on the other.

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 & 0 \\ \beta & \epsilon & a & h & g \\ \gamma & \zeta & h & b & f \\ \delta & \eta & g & f & c \end{vmatrix}$$

565 The sum of the products of each constituent of a column by the corresponding minor in another given column is zero. And the same is true if we read 'row' instead of 'column.' Thus, referring to the determinant in (555),

Taking the p^{th} and q^{th} columns, Taking the a and c rows,

$$a_pA_q+b_pB_q+\ldots+l_pL_q=0. \qquad a_1C_1+a_2C_2+\ldots+a_nC_n=0.$$

For in each case we have a determinant with two columns identical.

566 In any row or column the sum of the products of each constituent by its minor is the determinant itself. That is,

Taking the p^{th} column, Or taking the c row,

$$a_pA_p+b_pB_p+\ldots+l_pL_p=\Delta. \qquad c_1C_1+c_2C_2+\ldots+c_nC_n=\Delta.$$

567 The last equation may be expressed by $\Sigma c_pC_p=\Delta$.

Also, if (a_pc_q) express the determinant $\begin{vmatrix} a_p & a_q \\ c_p & c_q \end{vmatrix}$; then $\Sigma(a_pc_q)$ will represent the sum of all the determinants of the second order which can be formed by taking any two columns out of the a and c rows. The minor of (a_p, c_q) may be written (A_p, C_q), and signifies the determinant obtained by suppressing the two rows and two columns of a_p and c_q. Thus $\Delta=\Sigma(a_p, c_q)(A_p, C_q)$. And a similar notation when three or more rows and columns are selected.

568 *Analysis of a determinant.*

RULE.—*To resolve into its elements a determinant of the n^{th} order. Express it as the sum of n determinants of the $(n-1)^{th}$ order by* (560), *and repeat the process with each of the new determinants.*

EXAMPLE:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2\begin{vmatrix} b_3 & b_4 & b_1 \\ c_3 & c_4 & c_1 \\ d_3 & d_4 & d_1 \end{vmatrix} + a_3\begin{vmatrix} b_4 & b_1 & b_2 \\ c_4 & c_1 & c_2 \\ d_4 & d_1 & d_2 \end{vmatrix} - a_4\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

Again, $$\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = b_1\begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix} + b_2\begin{vmatrix} c_3 & c_1 \\ d_3 & d_1 \end{vmatrix} + b_3\begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix},$$

and so on. In the first series the determinants have alternately plus and minus signs, by the rule for cyclical interchanges (561), the order being even.

569 *Synthesis of a determinant.*

The process is facilitated by making use of two evident rules. Those constituents which belong to the row and column of a given constituent a, will be designated "a's constituents." Also, two pairs of constituents such as a_p, c_q and a_q, c_p, forming the corners of a rectangle, will be said to be "conjugate" to each other.

RULE I.—*No constituent will be found in the same term with one of its own constituents.*

RULE II.—*The conjugates of any two constituents a and b will be common to a's and b's constituents.*

Ex.—To write the following terms in the form of a determinant:

$$abcd+bfgl+f^2h^2+1edf+cghp+1ahr+elpr$$
$$-fhpr-ablr-ach^2-1fhg-bdf^2-efhl-cedp.$$

The determinant will be of the fourth order; and since every term must contain four constituents, the constituent 1 is supplied to make up the number in some of the terms. Select any term, as $abcd$, for the leading diagonal.

Now apply Rule I.,

a is not found with $e, f, f, g, p, 0$...(1). c is not found with $f, f, l, r, 1, 0$...(3).
b is not found with $e, h, h, p, 1, 0$...(2). d is not found with $g, h, h, l, r, 0$...(4).

Each constituent has $2(n-1)$, that is, 6 constituents belonging to it, since $n=4$. Assuming, therefore, that the above letters are the constituents of a, b, c, and d, and that there are no more, we supply a sixth zero constituent in each case.

Now apply Rule II.—The constituents common

to a and b are e, p; to a and c—f, f; to b and c—$1, 0$;
to a and d—$g, 0$; to b and d—$h, h, 0$; to c and d—$l, r, 0$.

$$\begin{vmatrix} a & e & f & g \\ p & b & 1 & h \\ f & 0 & c & r \\ 0 & h & l & d \end{vmatrix}$$

The determinant may now be formed. The diagonal being $abcd$; place e, p, the conjugates of a and b, either as in the diagram or transposed.

Then f and f, the conjugates of a and c, may be written.

1 and 0, the conjugates of b and c, must be placed as indicated, because 1 is one of p's constituents, since it is not found in any term with p, and must therefore be in the second row.

Similarly the places of g and 0, and of l and r, are assigned.

In the case of b and d we have $h, h, 0$ from which to choose the two conjugates, but we see that 0 is not one of them because that would assign two zero constituents to b, whereas b has but one, which is already placed.

By similar reasoning the ambiguity in selecting the conjugates l, r is removed.

The foregoing method is rigid in the case of a complete determinant having different constituents. It becomes uncertain when the zero constituents increase in number, and when several constituents are identical. But even then, in the majority of cases, it will soon afford a clue to the required arrangement.

570 *PRODUCT OF TWO DETERMINANTS OF THE n^{th} ORDER.*

$$\overset{(P)}{\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ \dots & \dots & \dots & \dots \\ l_1 & l_2 & \dots & l_n \end{vmatrix}} \overset{(Q)}{\begin{vmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \beta_1 & \beta_2 & \dots & \beta_n \\ \dots & \dots & \dots & \dots \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \end{vmatrix}} = \overset{(S)}{\begin{vmatrix} A_1 & A_2 & \dots & A_n \\ B_1 & B_2 & \dots & B_n \\ \dots & \dots & \dots & \dots \\ L_1 & L_2 & \dots & L_n \end{vmatrix}}$$

The values of A_1, B_1 ... L_1 in the first column of S are annexed. For the second column write b's in the place of a's. For the third column write c's, and so on.

$$\begin{cases} A_1 = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n \\ B_1 = a_1\beta_1 + a_2\beta_2 + \dots + a_n\beta_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ L_1 = a_1\lambda_1 + a_2\lambda_2 + \dots + a_n\lambda_n \end{cases}$$

For proof substitute the values of A_1, B_1, &c. in the determinant S, and then resolve S into the sum of a number of determinants by (562), and note the determinants which vanish through having identical columns.

RULE.—*To form the determinant S, which is the product of two determinants P and Q. First connect by plus signs the constituents in the rows of both the determinants P and Q.*

Now place the first row of P upon each row of Q in turn, and let each two constituents as they touch become products. This is the first column of S.

Perform the same operation upon Q with the second row of P to obtain the second column of S; and again with the third row of P to obtain the third column of S, and so on.

571 If the number of columns, both in P and Q, be n, and the number of rows r, and if n be $> r$, then the determinant S, found in the same way from P and Q, is equal to the sum of the $C(n, r)$ products of pairs of determinants obtained by taking any r columns out of P, and the corresponding r columns out of Q.

But if n be $< r$ the determinant S vanishes.

For in that case, in every one of the component determinants, there will be two columns identical.

572 The product of the determinants P and Q may be formed in four ways by changing the rows into columns in either or both P and Q.

573 Let the following system of n equations in $x_1 x_2 \dots x_n$ be transformed by substituting the accompanying values of the variables,

$$\begin{array}{ll} a_1x_1+a_2x_2+\dots+a_nx_n=0, & x_1=\alpha_1\xi_1+\alpha_2\xi_2+\dots+\alpha_n\xi_n, \\ b_1x_1+b_2x_2+\dots+b_nx_n=0, & x_2=\beta_1\xi_1+\beta_2\xi_2+\dots+\beta_n\xi_n, \\ \dots\dots\dots & \dots\dots\dots \\ l_1x_1+l_2x_2+\dots+l_nx_n=0; & x_n=\lambda_1\xi_1+\lambda_2\xi_2+\dots+\lambda_n\xi_n. \end{array}$$

The eliminant of the resulting equations in $\xi_1 \xi_2 \dots \xi_n$ is the determinant S in (570), and is therefore equal to the product of the determinants P and Q. The determinant Q is then termed the modulus of transformation.

574 *A Symmetrical determinant* is symmetrical about the leading diagonal. If the R's form the r^{th} row, and the K's the k^{th} row; then $R_k = K_r$ throughout a symmetrical determinant.

The square of a determinant is a symmetrical determinant.

575 *A Reciprocal determinant* has for its constituents the first minors of the original determinant, and is equal to its $n-1^{\text{th}}$ power; that is,

$$\begin{vmatrix} A_1 & \dots & A_n \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ L_1 & \dots & L_n \end{vmatrix} = \begin{vmatrix} a_1 & \dots & a_n \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ l_1 & \dots & l_n \end{vmatrix}^{n-1}$$

PROOF.—Multiply both sides of the equation by the original determinant (555). The constituents on the left side all vanish excepting the diagonal of Δ's.

576 *Partial and Complementary determinants.*

If r rows and the same number of columns be selected from a determinant, and if the rows be brought to the top, and the columns to the left side, without changing their order, then the elements common to the selected rows and columns form a Partial determinant of the order r, and the elements *not* found in any of those rows and columns form the Complementary determinant, its order being $n-r$.

Ex.—Let the selected rows from the determinant $(a_1b_2c_3d_4e_5)$ be the second, third, and fifth; and the selected columns be the third, fourth, and fifth. The original and the transformed determinants will be

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} \text{ and } \begin{vmatrix} b_3 & b_4 & b_5 & b_1 & b_2 \\ c_3 & c_4 & c_5 & c_1 & c_2 \\ e_3 & e_4 & e_5 & e_1 & e_2 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ d_3 & d_4 & d_5 & d_1 & d_2 \end{vmatrix}$$

The partial determinant of the third order is $(b_3c_4e_5)$, and its complementary of the second order is (a_1d_2).

The complete altered determinant is plus or minus, according as the permutations of the rows and columns are of the same or of different class. In the example they are of the same class, for there have been four transpositions of rows, and six of columns. Thus $(-1)^{10} = +1$ gives the sign of the altered determinant.

577 THEOREM.—A partial reciprocal determinant of the r^{th} order is equal to the product of the $r-1^{\text{th}}$ power of the original determinant, and the complementary of its corresponding partial determinant.

Take the last determinant for an example. Here $n=5$, $r=3$; and by the theorem,

$$\begin{vmatrix} B_3 & B_4 & B_5 \\ C_3 & C_4 & C_5 \\ E_3 & E_4 & E_5 \end{vmatrix} = \Delta^2 \begin{vmatrix} a_1 & a_2 \\ d_1 & d_2 \end{vmatrix}$$ where B, C, E are the respective minors.

PROOF.—Raise the Partial Reciprocal to the original order *five* without altering its value, by (564); and multiply it by Δ, with the rows and columns changed to correspond as in Ex. (576); thus, by (570), we have

$$\begin{vmatrix} B_3 & B_4 & B_5 & B_1 & B_2 \\ C_3 & C_4 & C_5 & C_1 & C_2 \\ E_3 & E_4 & E_5 & E_1 & E_2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} b_3 & b_4 & b_5 & b_1 & b_2 \\ c_3 & c_4 & c_5 & c_1 & c_2 \\ e_3 & e_4 & e_5 & e_1 & e_2 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ d_3 & d_4 & d_5 & d_1 & d_2 \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 & b_1 & b_2 \\ 0 & \Delta & 0 & c_1 & c_2 \\ 0 & 0 & \Delta & e_1 & e_2 \\ 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & d_1 & d_2 \end{vmatrix} = \Delta^3 \begin{vmatrix} a_1 & a_2 \\ d_1 & d_2 \end{vmatrix}$$

578 The product of the differences between every pair of n quantities $a_1, a_2 \dots a_n$, ...

$$\left.\begin{array}{r} (a_1-a_2)(a_1-a_3)(a_1-a_4)\dots(a_1-a_n) \\ \times(a_2-a_3)(a_2-a_4)\dots(a_2-a_n) \\ \times(a_3-a_4)\dots(a_3-a_n) \\ \dots\dots\dots\dots\dots\dots \\ \times(a_{n-1}-a_n) \end{array}\right\} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \dots & a_n^{n-1} \end{vmatrix}$$

PROOF.—The determinant vanishes when any two of the quantities are equal. Therefore it is divisible by each of the factors on the left; therefore by their product. And the quotient is seen to be unity, for both sides of the equation are of the same degree; viz., $\frac{1}{2}n(n-1)$.

579 The product of the *squares* of the differences of the same n quantities $\Bigg\} = \begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{vmatrix}$

PROOF.—Square the determinant in (578), and write s_r for the sum of the r^{th} powers of the roots.

580 With the same meaning for $s_1, s_2 \dots$, the same determinant taken of an order r, less than n, is equal to the sum of the products of the squares of the differences of r of the n quantities taken in every possible way; that is, in $C(n, r)$ ways.

Ex.:
$$\begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} = (a_1-a_2)^2+(a_1-a_3)^2+\&c. \equiv \Sigma(a_1-a_2)^2,$$
$$\begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = \Sigma(a_1-a_2)^2(a_1-a_3)^2(a_2-a_3)^2.$$

The next determinant in order
$$= \Sigma(a_1-a_2)^2(a_1-a_3)^2(a_1-a_4)^2(a_2-a_3)^2(a_2-a_4)^2(a_3-a_4)^2.$$

And so on until the equation (579) is reached.

Proved by substituting the values of $s_1, s_2 \dots$ &c., and resolving the determinant into its partial determinants by (571).

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581 The quotient of

$$\frac{a_0x^m+a_1x^{m-1}+\dots+a_rx^{m-r}+\dots}{b_0x^n+b_1x^{n-1}+\dots+b_rx^{n-r}+\dots}$$

is given by the formula

$$q_0x^{m-n}+q_1x^{m-n-1}+\dots+q_rx^{m-n-r}+\dots,$$

where
$$q_r = \frac{1}{b_0^{r+1}}\begin{vmatrix} b_0 & 0 & 0 & \dots & a_0 \\ b_1 & b_0 & 0 & \dots & a_1 \\ b_2 & b_1 & b_0 & \dots & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ b_r & b_{r-1} & b_{r-2} & \dots\ b_1 & a_r \end{vmatrix}.$$

Proved by Induction.

ELIMINATION.

582 ***Solution of n linear equations in n variables.***

The equations and the values of the variables are arranged below:

$$\begin{array}{ll} a_1x_1+a_2x_2+\ldots+a_nx_n = \xi_1, & x_1\Delta = A_1\xi_1+B_1\xi_2+\ldots+L_1\xi_n \\ b_1x_1+b_2x_2+\ldots+b_nx_n = \xi_2, & x_2\Delta = A_2\xi_1+B_2\xi_2+\ldots+L_2\xi_n \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ l_1x_1+l_2x_2+\ldots+l_nx_n = \xi_n, & x_n\Delta = A_n\xi_1+B_n\xi_2+\ldots+L_n\xi_n \end{array}$$

where Δ is the determinant annexed, and A_1, B_1, &c. are its first minors.

$$\begin{vmatrix} a_1 & \ldots & a_n \\ \ldots & \ldots & \ldots \\ l_1 & \ldots & l_n \end{vmatrix}$$

To find the value of one of the unknowns x_r.

RULE.—*Multiply the equations respectively by the minors of the r^{th} column, and add the results. x_r will be equal to the fraction whose numerator is the determinant Δ, with its r^{th} column replaced by $\xi_1, \xi_2 \ldots \xi_n$, and whose denominator is Δ itself.*

583 If $\xi_1, \xi_2 \ldots \xi_n$ and Δ all vanish, then $x_1, x_2 \ldots x_n$ are in the ratios of the minors of any row of the determinant Δ. For example, in the ratios $C_1 : C_2 : C_3 : \ldots : C_n$.

The eliminant of the given equations is now $\Delta = 0$.

584 ***Orthogonal Transformation.***

If the two sets of variables in the n equations (582) be connected by the relation

$$x_1^2+x_2^2+\ldots+x_n^2 = \xi_1^2+\xi_2^2+\ldots+\xi_n^2 \quad \ldots\ldots (1),$$

then the changing from one set of variables to the other, by substituting the values of the ξ's in terms of the x's in any function of the former, or *vice versâ*, is called orthogonal transformation.

When equation (1) is satisfied, two results follow.

I. *The determinant* $\Delta = \pm 1$.

II. *Each of the constituents of Δ is equal to the corresponding minor, or else to minus that minor according as Δ is positive or negative.*

PROOF.—Substitute the values of $\xi_1, \xi_2 \dots \xi_n$ in terms of $x_1, x_2 \dots x_n$ in equation (1), and equate coefficients of the squares and products of the new variables. We get the n^2 equations

$$\left.\begin{array}{l} a_1^2 + b_1^2 \;+ = 1 \\ a_1a_2 + b_1b_2 + = 0 \\ a_1a_3 + b_1b_3 + = 0 \\ \dots\dots\dots\dots \\ a_1a_n + b_1b_n + = 0 \end{array}\right\} \quad \left.\begin{array}{l} a_2a_1 + b_2b_1 + = 0 \\ a_2^2 + b_2^2 \;+ = 1 \\ a_2a_3 + b_2b_3 + = 0 \\ \dots\dots\dots\dots \\ a_2a_n + b_2b_n + = 0 \end{array}\right\} \quad \left.\begin{array}{l} a_3a_1 + b_3b_1 + = 0 \\ a_3a_2 + b_3b_2 + = 0 \\ a_3^2 + b_3^2 \;+ = 1 \\ \dots\dots\dots\dots \\ a_3a_n + b_3b_n + = 0 \end{array}\right\}$$

Also $\Delta = \begin{vmatrix} a_1b_1 \dots l_1 \\ a_2b_2 \dots l_2 \\ a_3b_3 \dots l_3 \\ \dots\dots \\ a_nb_n \dots l_n \end{vmatrix}$ Form the square of the determinant Δ by the rule (570), and these equations show that the product is a determinant in which the only constituents that do not vanish constitute a diagonal of 'ones.' Therefore

$$\Delta^2 = 1 \text{ and } \Delta = \pm 1.$$

Again, solving the first set of equations for a_1 (writing a_1^2 as a_1a_1, &c.), the second set for a_2, the third for a_3, and so on, we have, by (582), the results annexed; which proves the second proposition.

$$\left\{\begin{array}{l} a_1\Delta = A_1 \;+A_20+A_30+ = A_1 \\ a_2\Delta = A_10+A_2 \;+A_30+ = A_2 \\ a_3\Delta = A_10+A_20+A_3 \;+ = A_3 \\ \qquad \&c. \qquad \&c. \end{array}\right.$$

585 *Theorem.*—The $n-2^{\text{th}}$ power of a determinant of the n^{th} order multiplied by any constituent is equal to the corresponding minor of the reciprocal determinant.

PROOF.—Let ρ be the reciprocal determinant of Δ, and β_r the minor of B_r in ρ. Write the transformed equations (582) for the x's in terms of the ξ's, and solve them for ξ_2. Then equate the coefficient of x_r in the result with its coefficient in the original value of ξ_2.

Thus $\rho\xi_2 = \Delta(\beta_1x_1 + \dots + \beta_rx_r + \dots)$, and $\xi_2 = b_1x_1 + \dots + b_rx_r + \dots$;

$$\therefore \; \Delta\beta_r = \rho b_r = \Delta^{n-1}b_r \text{ by (575)}; \quad \therefore \; \beta_r = \Delta^{n-2}b_r.$$

586 To eliminate x from the two equations

$$ax^m + bx^{m-1} + cx^{m-2} + \dots = 0 \quad \dots\dots\dots\dots (1),$$
$$a'x^n + b'x^{n-1} + c'x^{n-2} + \dots = 0 \quad \dots\dots\dots\dots (2).$$

If it is desired that the equation should be homogeneous in x and y; put $\frac{x}{y}$ instead of x, and clear of fractions. The following methods will still be applicable.

I. ***Bezout's Method.***—Suppose $m > n$.

RULE.—*Bring the equations to the same degree by multiplying* (2) *by* x^{m-n}. *Then multiply* (1) *by* a', *and* (2) *by* a, *and subtract.*

Again, multiply (1) *by* $a'x+b'$, *and* (2) *by* $(ax+b)$, *and subtract.*

Again, multiply (1) *by* $a'x^2+b'x+c'$, *and* (2) *by* (ax^2+bx+c), *and subtract, and so on until* n *equations have been obtained. Each will be of the degree* $m-1$.

Write under these the $m-n$ *equations obtained by multiplying* (2) *successively by* x. *The eliminant of the* m *equations is the result required.*

Ex.—Let the equations be $\begin{cases} ax^5+bx^4+cx^3+dx^2+ex+f=0, \\ a'x^3+b'x^2+c'x+d' = 0. \end{cases}$

The five equations obtained by the method, and their eliminant, by (583), are, writing capital letters for the functions of a, b, c, d, e, f,

$$\left.\begin{array}{r} A_1x^4+B_1x^3+C_1x^2+D_1x+E_1=0 \\ A_2x^4+B_2x^3+C_2x^2+D_2x+E_2=0 \\ A_3x^4+B_3x^3+C_3x^2+D_3x+E_3=0 \\ a'x^4+b'x^3+c'x^2+d'x \quad =0 \\ a'x^3+b'x^2+c'x+d' =0 \end{array}\right\} \quad \text{and} \quad \begin{vmatrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 \\ A_3 & B_3 & C_3 & D_3 & E_3 \\ a' & b' & c' & d' & 0 \\ 0 & a' & b' & c' & d' \end{vmatrix} = 0.$$

Should the equations be of the same degree, the eliminant will be a symmetrical determinant.

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587 II. *Sylvester's Dialytic Method.*

RULE.—*Multiply equation* (1) *successively by* x, $n-1$ *times; and equation* (2) $m-1$ *times; and eliminate* x *from the* $m+n$ *resulting equations.*

Ex.—To eliminate x from $\left.\begin{array}{r} ax^3+bx^2+cx+d=0 \\ px+qx+r=0 \end{array}\right\}$.

The $m+n$ equations and their eliminant are

$$\left.\begin{array}{r} px^2+qx+r=0 \\ px^3+qx^2+rx \quad =0 \\ px^4+qx^3+rx^2 \quad =0 \\ ax^3+bx^2+cx+d=0 \\ ax^4+bx^3+cx^2+dx \quad =0 \end{array}\right\} \quad \text{and} \quad \begin{vmatrix} 0 & 0 & p & q & r \\ 0 & p & q & r & 0 \\ p & q & r & 0 & 0 \\ 0 & a & b & c & d \\ a & b & c & d & 0 \end{vmatrix} = 0.$$

588 III. *Method of elimination by Symmetrical Functions.*

Divide the two equations in (586) respectively by the coefficients of their first terms, thus reducing them to the forms

$$f(x) \equiv x^m + p_1 x^{m-1} + \dots + p_m = 0,$$
$$\phi(x) \equiv x^n + q_1 x^{n-1} + \dots + q_n = 0.$$

RULE.—*Let $a, b, c \dots$ represent the roots of $f(x)$. Form the equation $\phi(a)\,\phi(b)\,\phi(c) \dots = 0$. This will contain symmetrical functions only of the roots $a, b, c \dots$.*

Express these functions in terms of $p_1, p_2 \dots$ by (538), &c., and the equation becomes the eliminant.

Reason of the rule.—The eliminant is the condition for a common root of the two equations. That root must make one of the factors $\phi(a), \phi(b) \dots$ vanish, and therefore it makes their product vanish.

589 The eliminant expressed in terms of the roots $a, b, c \dots$ of $f(x)$, and the roots $\alpha, \beta, \gamma \dots$ of $\phi(x)$, will be

$$(a-\alpha)(a-\beta)(a-\gamma) \dots (b-\alpha)(b-\beta)(b-\gamma) \dots \&c.,$$

being the product of all possible differences between a root of one equation and a root of another.

590 The eliminant is a homogeneous function of the coefficients of either equation, being of the n^{th} degree in the coefficients of $f(x)$, and of the m^{th} degree in the coefficients of $\phi(x)$.

591 The sum of the suffixes of p and q in each term of the eliminant $= mn$. Also, if p, q contain z; if p_2, q_2 contain z^2; if p_3, q_3 contain z^3, and so on, the eliminant will contain z^{mn}.

Proved by the fact that p_r is a homogeneous function of r dimensions of the roots $a, b, c \dots$, by (406).

592 If the two equations involve x and y, the elimination may be conducted with respect to x; and y will be contained in the coefficients $p_1, p_2 \dots, q_1, q_2 \dots$.

593 *Elimination by the Method of Highest Common Factor.*

Let two algebraical equations in x and y be represented by $A = 0$ and $B = 0$.

It is required to eliminate x.

Arrange A and B according to descending powers of x, and, having rejected any factor which is a function of y only, proceed to find the Highest Common Factor of A and B.

The process may be exhibited as follows:

$$\left.\begin{aligned} c_1A &= q_1B + r_1R_1 \\ c_2B &= q_2R_1 + r_2R_2 \\ c_3R &= q_3R_2 + r_3R_3 \\ c_4R &= q_4R_3 + r_4 \end{aligned}\right\}$$

c_1, c_2, c_3, c_4 are the multipliers required at each stage in order to avoid fractional quotients; and these must be constants or functions of y only.

q_1, q_2, q_3, q_4 are the successive quotients.

$r_1R_1, r_2R_2, r_3R_3, r_4$ are the successive remainders; r_1, r_2, r_3, r_4 being functions of y only.

The process terminates as soon as a remainder is obtained which is a function of y only; r_4 is here supposed to be such a remainder.

Now, the simplest factors having been taken for c_1, c_2, c_3, c_4, we see that

$$\left.\begin{array}{llll} 1 \text{ is the H. C. F. of} & c_1 & \text{and } r_1 \\ d_2 \quad ,, \quad ,, & c_1 & \text{and } r_2 \\ d_3 \quad ,, \quad ,, & \dfrac{c_1c_2}{d_2} & \text{and } r_3 \\ d_4 \quad ,, \quad ,, & \dfrac{c_1c_2c_3}{d_2d_3} & \text{and } r_4 \end{array}\right\}$$

The values of x and y, which satisfy simultaneously the equations $A=0$ and $B=0$, are those obtained by the four pairs of simultaneous equations following:

$$\left.\begin{aligned} r_1 &= 0 \text{ and } B = 0 \ldots\ldots\ldots (1) \\ \frac{r_2}{d_2} &= 0 \text{ and } R_1 = 0 \ldots\ldots\ldots (2) \\ \frac{r_3}{d_3} &= 0 \text{ and } R_2 = 0 \ldots\ldots\ldots (3) \\ \frac{r_4}{d_4} &= 0 \text{ and } R_3 = 0 \ldots\ldots\ldots (4) \end{aligned}\right\}$$

The final equation in y, which gives all admissible values, is

$$\frac{r_1r_2r_3r_4}{d_2d_3d_4} = 0.$$

If it should happen that the remainder r_4 is zero, the simultaneous equations (1), (2), (3), and (4) reduce to

$$r_1 = 0 \text{ and } \frac{B}{R_3} = 0; \quad \frac{r_2}{d_2} = 0 \text{ and } \frac{R_1}{R_3} = 0; \quad \frac{r_3}{d_3} = 0 \text{ and } \frac{R_2}{R_3} = 0.$$

594 To find infinite values of x or y which satisfy the given equations.

Put $x = \frac{1}{z}$. Clear of fractions, and make $z = 0$.

If the two resulting equations in y have any common roots, such roots, together with $x = \infty$, satisfy simultaneously the equations proposed.

Similarly we may put $y = \frac{1}{z}$.

PLANE TRIGONOMETRY.

ANGULAR MEASUREMENT.

600 The unit of Circular measure is the angle at the centre of a circle which subtends an arc equal to the radius. Hence

601 Circular measure of an angle $= \dfrac{\text{arc}}{\text{radius}}$.

602 Circular measure of two right angles $= 3{\cdot}14159\ldots \equiv \pi$.

603 The unit of Centesimal measure is a Grade, and is the one-hundredth part of a right angle.

604 The unit of Sexagesimal measure is a Degree, and is the one-sixtieth part of a right angle.

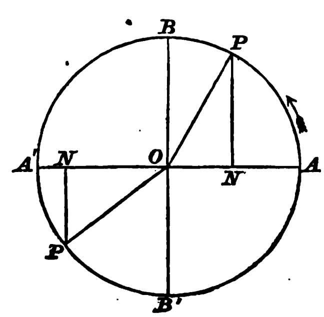
To change degrees into grades, or circular measure, or *vice versâ*, employ one of the three equations included in

605
$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi},$$

where D, G, and C are respectively the numbers of degrees, grades, and units of circular measure in the angle considered.

TRIGONOMETRICAL RATIOS.

606 Let OA be fixed, and let the revolving line OP describe a circle round O. Draw PN always perpendicular to AA'. Then, in all positions of OP,



$\dfrac{PN}{OP} =$ the sine of the angle AOP,

$\dfrac{ON}{OP} =$ the cosine of the angle AOP,

$\dfrac{PN}{ON} =$ the tangent of the angle AOP.

607 If P be *above* the line AA', sin AOP is positive.
If P be *below* the line AA', sin AOP is negative.

608 If P lies to the *right* of BB', cos AOP is positive.
If P lies to the *left* of BB', cos AOP is negative.

609 Note, that by the angle AOP is meant the angle through which OP has revolved from OA, its initial position; and this angle of revolution may have any magnitude. If the revolution takes place in the opposite direction, the angle described is reckoned negative.

610 The secant of an angle is the reciprocal of its cosine, or
$$\cos A \sec A = 1.$$

611 The cosecant of an angle is the reciprocal of its sine, or
$$\sin A \operatorname{cosec} A = 1.$$

612 The cotangent of an angle is the reciprocal of its tangent, or
$$\tan A \cot A = 1.$$

Relations between the trigonometrical functions of the same angle.

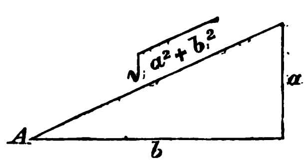
613 $$\sin^2 A + \cos^2 A = 1.$$ [I. 47

614 $$\sec^2 A = 1 + \tan^2 A.$$

615 $$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

616 $$\tan A = \frac{\sin A}{\cos A}.$$ [606

If $\tan A = \dfrac{a}{b}$.



617 $$\sin A = \frac{a}{\sqrt{a^2+b^2}},$$

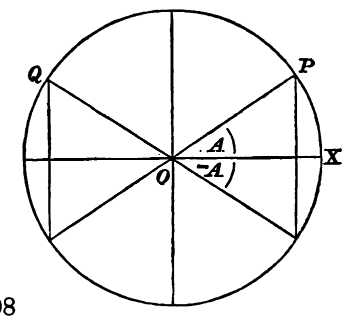
$$\cos A = \frac{b}{\sqrt{a^2+b^2}}.$$ [606

618 $$\sin A = \frac{\tan A}{\sqrt{1+\tan^2 A}}, \quad \cos A = \frac{1}{\sqrt{1+\tan^2 A}}$$ [617

619 The Complement of A is $= 90°-A$.

620 The Supplement of A is $= 180°-A$.

621 $\sin(90°-A) = \cos A$,
$\tan(90°-A) = \cot A$,
$\sec(90°-A) = \operatorname{cosec} A$.



622 $\sin(180°-A) = \sin A$,
$\cos(180°-A) = -\cos A$,
$\tan(180°-A) = -\tan A$.

In the figure

$\angle QOX = 180°-A$. [607, 608

623 $\sin(-A) = -\sin A$.

624 $\cos(-A) = \cos A$.

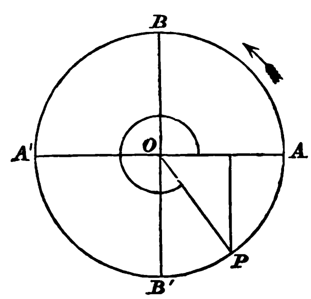
By Fig., and (607), (608).

The secant, cosecant, and cotangent of $180°-A$, and of $-A$, will follow the same rule as their reciprocals, the cosine, sine, and tangent. [610—612

625 To reduce any ratio of an angle greater than 90° to the ratio of an angle less than 90°.

Rule.—Determine the sign of the ratio by the rules (607), and then substitute for the given angle the *acute* angle formed by its two bounding lines, produced if necessary.

Ex.—To find all the ratios of 660°. Measuring 300° (= 660°−360°) round the circle from A to P, we find the acute angle AOP to be 60°, and P lies *below* AA, and to the *right* of BB'.



Therefore

$\sin 660° = -\sin 60° = -\dfrac{\sqrt{3}}{2}$,

$\cos 660° = \cos 60° = \frac{1}{2}$,

and from the sine and cosine all the remaining ratios may be found by (613—616).

Inverse Notation.—The angle whose sine is x is denoted by $\sin^{-1} x$.

626 All the angles which have a given sine, cosine, or tangent, are given by the formulæ

$$\sin^{-1} x = n\pi + (-1)^n \theta \quad \text{.............. (1)},$$
$$\cos^{-1} x = 2n\pi \pm \theta \quad \text{...................... (2)},$$
$$\tan^{-1} x = n\pi + \theta \quad \text{...................... (3)}.$$

In these formulæ θ is any angle which has x for its sine, cosine, or tangent respectively, and n is any integer.

Cosec$^{-1} x$, sec$^{-1} x$, cot$^{-1} x$ have similar general values, by (610—612).

These formulæ are verified by taking A, in Fig. 622, for θ, and making n an odd or even integer successively.

FORMULÆ INVOLVING TWO ANGLES, AND MULTIPLE ANGLES.

627 $\sin(A+B) = \sin A \cos B + \cos A \sin B,$

628 $\sin(A-B) = \sin A \cos B - \cos A \sin B,$

629 $\cos(A+B) = \cos A \cos B - \sin A \sin B,$

630 $\cos(A-B) = \cos A \cos B + \sin A \sin B.$

Proofs of (627) to (630).—By (700) and (701), we have

$$\sin C = \sin A \cos B + \cos A \sin B,$$

and $$\sin C = \sin(A+B), \text{ by (622).}$$

To obtain $\sin(A-B)$ change the sign of B in (627), and employ (623), (624), $\cos(A+B) = \sin\{(90°-A)-B\}$, by (621).

Expand by (628), and use (621), (623), (624). For $\cos(A-B)$ change the sign of B in (629).

631 $$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}.$$

632 $$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}.$$

633 $$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}.$$

634 $$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

Obtained from (627—630).

635 $\sin 2A = 2\sin A\cos A.$ [627. Put $B=A$

636 $\cos 2A = \cos^2 A - \sin^2 A,$

637 $= 2\cos^2 A - 1,$

638 $= 1 - 2\sin^2 A.$ [629, 613

639 $2\cos^2 A = 1 + \cos 2A.$ [637

640 $2\sin^2 A = 1 - \cos 2A.$ [638

641 $\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}.$ [640

642 $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}.$ [639

643 $\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A}.$ [641, 642, 613

646 $\cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}.\quad \sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}.$ [643, 613

648 $\cos A = \frac{1}{1+\tan A\tan\frac{A}{2}}.$

649 $\sin\left(45^\circ+\frac{A}{2}\right) = \cos\left(45^\circ-\frac{A}{2}\right) = \sqrt{\frac{1+\sin A}{2}}.$ [641

650 $\cos\left(45^\circ+\frac{A}{2}\right) = \sin\left(45^\circ-\frac{A}{2}\right) = \sqrt{\frac{1-\sin A}{2}}.$ [642

651 $\tan\left(45^\circ+\frac{A}{2}\right) = \sqrt{\frac{1+\sin A}{1-\sin A}} = \frac{1+\sin A}{\cos A} = \frac{\cos A}{1-\sin A}.$

652 $\tan 2A = \frac{2\tan A}{1-\tan^2 A}.$ [631. Put $B=A$

653 $\cot 2A = \frac{\cot^2 A - 1}{2\cot A}.$

654 $$\tan(45^\circ + A) = \frac{1+\tan A}{1-\tan A}.$$

655 $$\tan(45^\circ - A) = \frac{1-\tan A}{1+\tan A}.$$ [631, 632

656 $$\sin 3A = 3\sin A - 4\sin^3 A.$$

657 $$\cos 3A = 4\cos^3 A - 3\cos A.$$

658 $$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

By putting $B=2A$ in (627), (629), and (631).

659 $$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$$

660 $$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$

From (627), &c.

661 $$\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A}.$$

Proved by squaring.

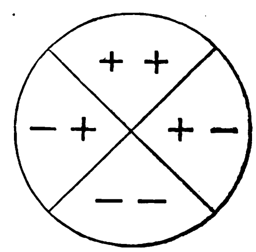
662 $$\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A}.$$

663 $$\sin\frac{A}{2} = \frac{1}{2}\left\{\sqrt{1+\sin A} - \sqrt{1-\sin A}\right\}.$$

664 $$\cos\frac{A}{2} = \frac{1}{2}\left\{\sqrt{1+\sin A} + \sqrt{1-\sin A}\right\},$$

when $\frac{A}{2}$ lies between -45° and $+45^\circ$.

665 In the accompanying diagram the signs exhibited in each quadrant are the signs to be prefixed to the two surds in the value of $\sin\frac{A}{2}$ according to the quadrant in which $\frac{A}{2}$ lies.



For $\cos\frac{A}{2}$ change the second sign.

Proved by examining the changes of sign in (661) and (662) by (607).

666 $\sin(A+B)+\sin(A-B) = 2\sin A\cos B.$

667 $\sin(A+B)-\sin(A-B) = 2\cos A\sin B.$

668 $\cos(A+B)+\cos(A-B) = 2\cos A\cos B.$

669 $\cos(A-B)-\cos(A+B) = 2\sin A\sin B.$

[627—630

670 $$\sin A+\sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

671 $$\sin A-\sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

672 $$\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

673 $$\cos B-\cos A = 2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

Obtained by changing A into $\frac{A+B}{2}$, and B into $\frac{A-B}{2}$, in (666—669).

It is advantageous to commit the foregoing formulæ to memory, in words, thus—

sin sum + sin difference = 2 sin cos,
sin sum − sin difference = 2 cos sin,
cos sum + cos difference = 2 cos cos,
cos difference − cos sum = 2 sin sin.

sin first + sin second = 2 sin half sum cos half difference,
sin first − sin second = 2 cos half sum sin half difference,
cos first + cos second = 2 cos half sum cos half difference,
cos second − cos first = 2 sin half sum sin half difference.

---

674 $\sin(A+B+C)$
$$= \sin A\cos B\cos C+\sin B\cos C\cos A + \sin C\cos A\cos B-\sin A\sin B\sin C.$$

675 $\cos(A+B+C)$
$$= \cos A\cos B\cos C-\cos A\sin B\sin C - \cos B\sin C\sin A-\cos C\sin A\sin B.$$

676 $\tan(A+B+C)$
$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}.$$

Put $B+C$ for B in (627), (629), and (631).

If $A+B+C=180^\circ$,

677 $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

$\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$.

678 $\cos A+\cos B+\cos C=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+1$.

$\cos A+\cos B-\cos C=4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}-1$.

679 $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

680 $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

681 $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

682 $\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C-1$.

General formulæ, including the foregoing, obtained by applying (666—673).

If $A+B+C=\pi$, and n be any integer,

683 $4\sin\frac{nA}{2}\sin\frac{nB}{2}\sin\frac{nC}{2}$

$$=\sin\left(\frac{n\pi}{2}-nA\right)+\sin\left(\frac{n\pi}{2}-nB\right)+\sin\left(\frac{n\pi}{2}-nC\right)-\sin\frac{n\pi}{2}.$$

684 $4\cos\frac{nA}{2}\cos\frac{nB}{2}\cos\frac{nC}{2}$

$$=\cos\left(\frac{n\pi}{2}-nA\right)+\cos\left(\frac{n\pi}{2}-nB\right)+\cos\left(\frac{n\pi}{2}-nC\right)+\cos\frac{n\pi}{2}.$$

If $A+B+C=0$,

685 $4\sin\frac{nA}{2}\sin\frac{nB}{2}\sin\frac{nC}{2}=-\sin nA-\sin nB-\sin nC$.

686 $4\cos\frac{nA}{2}\cos\frac{nB}{2}\cos\frac{nC}{2}=\cos nA+\cos nB+\cos nC+1$.

RULE.—*If, in formulæ* (683) *to* (686), *two factors on the left be changed by writing sin for cos, or cos for sin; then, on the right side, change the signs of those terms which do not contain the angles of the altered factors.*

Thus, from (683), we obtain

687 $$4\sin\frac{nA}{2}\cos\frac{nB}{2}\cos\frac{nC}{2}$$
$$= -\sin\left(\frac{n\pi}{2}-nA\right)+\sin\left(\frac{n\pi}{2}-nB\right)+\sin\left(\frac{n\pi}{2}-nC\right)+\sin\frac{n\pi}{2}.$$

A Formula for the construction of Tables of sines, cosines, &c.—

688 $$\sin(n+1)\,a-\sin na = \sin na-\sin(n-1)\,a-k\sin na,$$
where $a = 10''$, and $k = 2(1-\cos a) = \cdot 0000000023504$.

689 Formulæ for verifying the tables—
$$\sin A+\sin(72^\circ+A)-\sin(72^\circ-A) = \sin(36^\circ+A)-\sin(36^\circ-A),$$
$$\cos A+\cos(72^\circ+A)+\cos(72^\circ-A) = \cos(36^\circ+A)+\cos(36^\circ-A),$$
$$\sin(60^\circ+A)-\sin(60^\circ-A) = \sin A.$$

RATIOS OF CERTAIN ANGLES.

690 $$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}},\quad \tan 45^\circ = 1.$$

691 $$\sin 60^\circ = \frac{\sqrt{3}}{2},\quad \cos 60^\circ = \frac{1}{2},\quad \tan 60^\circ = \sqrt{3}.$$

692 $$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}},\quad \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}},$$
$$\left.\begin{array}{l}\tan 15^\circ = 2-\sqrt{3}\\ \cot 15^\circ = 2+\sqrt{3}\end{array}\right\}.$$

693 $$\sin 18^\circ = \frac{\sqrt{5}-1}{4},\quad \cos 18^\circ = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}},$$
$$\tan 18^\circ = \sqrt{\frac{5-2\sqrt{5}}{5}}.$$

694 $$\sin 54^\circ = \frac{\sqrt{5}+1}{4},\quad \cos 54^\circ = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}},$$
$$\tan 54^\circ = \sqrt{\frac{5+2\sqrt{5}}{5}}.$$

695 By taking the complements of these angles, the same table gives the ratios of 30°, 75°, 72°, and 36°.

696 NOTE.—sin 15° is obtained from sin (45°−30°), expanded by (628).

697 sin 18° from the equation $\sin 2x = \cos 3x$, where $x = 18°$.

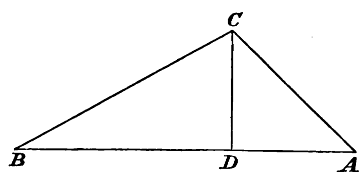
698 sin 54° from $\sin 3x = 3\sin x - 4\sin^3 x$, where $x = 18°$.

699 And the ratios of various angles may be obtained by taking the sum, difference, or some multiple of the angles in the table, and making use of known formulæ. Thus

$$12° = 30° - 18°,\quad 7\tfrac{1}{2}° = \frac{15°}{2},\quad \&c.\ \&c.$$

PROPERTIES OF THE TRIANGLE.

700 $c = a\cos B + b\cos A$.



701 $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

702 $a^2 = b^2 + c^2 - 2bc\cos A$.

[By Euc. II. 5 & 6.

703 $$\cos A = \frac{b^2+c^2-a^2}{2bc}.$$

If $s \equiv \dfrac{a+b+c}{2}$, and Δ denote the area ABC,

704 $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

[641, 642, 703, 11, 12

705 $$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

706 $$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$ [635, 704

707 $$\Delta = \frac{bc}{2}\sin A = \sqrt{s(s-a)(s-b)(s-c)}.$$ [707, 706

708 $$= \tfrac{1}{4}\sqrt{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}.$$

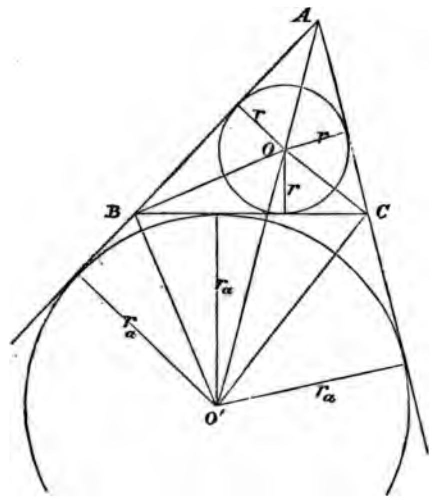
The Triangle and Circle.

Let

r = radius of inscribed circle.

r_a = radius of escribed circle touching the side a.

R = radius of circumscribing circle.



709 $$r = \frac{\Delta}{s}.$$

[From Fig., $\Delta = \frac{ra}{2} + \frac{rb}{2} + \frac{rc}{2}$.

710 $$r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}.$$

[By $a = r \cot \frac{B}{2} + r \cot \frac{C}{2}$.

711 $$r_a = \frac{\Delta}{s-a}.$$ [By $\Delta = \frac{r_a b}{2} + \frac{r_a c}{2} - \frac{r_a a}{2}$.

712 $$r_a = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}.$$ [From $a = r_a \tan \frac{B}{2} + r_a \tan \frac{C}{2}$.

---

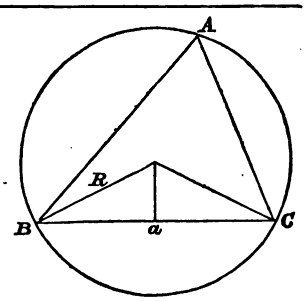
713 $$R = \frac{a}{2 \sin A} = \frac{abc}{4\Delta}.$$

[By (III 20) and (706).

715

$$= \tfrac{1}{4}\sqrt{\left\{(b+c)^2 \sec^2 \frac{A}{2} + (b-c)^2 \operatorname{cosec}^2 \frac{A}{2}\right\}}$$

[702



Distance between the centres of inscribed and circumscribed circles

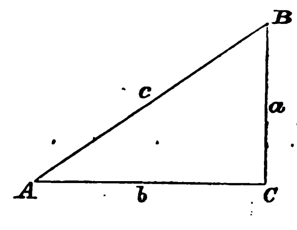
716 $$= \sqrt{R^2 - 2Rr}.$$ [936

Radius of circle touching b, c and the inscribed circle

717 $$r' = r \tan^2 \tfrac{1}{4}(B+C).$$ [By $\sin \frac{A}{2} = \frac{r-r'}{r+r'}$.

SOLUTION OF TRIANGLES.

Right-angled triangles are solved by the formulæ



718 $c^2 = a^2+b^2;$

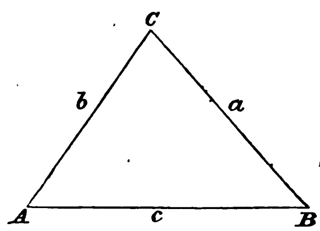
719 $\begin{cases} a = c \sin A, \\ b = c \cos A, \\ a = b \tan A, \end{cases}$
&c.

Scalene Triangles.

720 CASE I.—The equation

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad [701$$

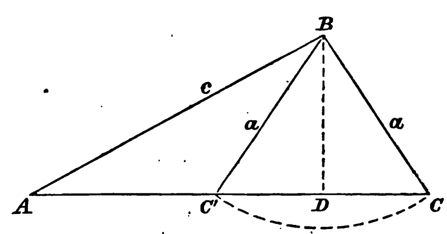
will determine any one of the four quantities A, B, a, b when the remaining three are known.



721 *The Ambiguous Case.*

When, in Case I., two sides and an acute angle opposite to one of them are given, we have, from the figure,

$$\sin C = \frac{c \sin A}{a}.$$



Then C and $180°-C$ are the values of C and C', by (622).

Also $\qquad b = c \cos A \pm \sqrt{a^2-c^2 \sin^2 A},$

because $\qquad = AD \pm DC.$

722 When an angle B is to be determined from the equation

$$\sin B = \frac{b}{a} \sin A,$$

and $\frac{b}{a}$ is a small fraction; the circular measure of B may be approximated to by putting $\sin(B+C)$ for $\sin A$, and using theorem (796).

723 CASE II.—When two sides b, c and the included angle A are known, the third side a is given by the formula

$$a^2 = b^2+c^2-2bc\cos A, \quad [702$$

when logarithms are not used.

Otherwise, employ the following formula with logarithms,

724 $$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

Obtained from $\frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C}$ (701), and then applying (670) and (671).

$\frac{B-C}{2}$ having been found from the above equation, and $\frac{B+C}{2}$ being equal to $90° - \frac{A}{2}$, we have

725 $$B = \frac{B+C}{2} + \frac{B-C}{2}, \qquad C = \frac{B+C}{2} - \frac{B-C}{2}.$$

B and C having been determined, a can be found by Case I.

726 If the logarithms of b and c are known, the trouble of taking out $\log(b-c)$ and $\log(b+c)$ may be avoided by employing the subsidiary angle $\theta = \tan^{-1}\frac{b}{c}$, and the formula

727 $$\tan\tfrac{1}{2}(B-C) = \tan\left(\theta - \frac{\pi}{4}\right)\cot\frac{A}{2}. \quad [655$$

Or else the subsidiary angle $\theta = \cos^{-1}\frac{c}{b}$, and the formula

728 $$\tan\tfrac{1}{2}(B-C) = \tan^2\frac{\theta}{2}\cot\frac{A}{2}. \quad [643$$

If a be required without calculating the angles B and C, we may use the formula

729 $$a = \frac{(b+c)\sin\frac{A}{2}}{\cos\frac{1}{2}(B-C)}.$$

From the figure in 940, by drawing a perpendicular from B to EC produced.

730 If a be required in terms of b, c, and A alone, and in a form adapted to logarithmic computation, employ the subsidiary angle

$$\theta = \sin^{-1}\left(\frac{4bc}{(b+c)^2}\cos^2\frac{A}{2}\right),$$

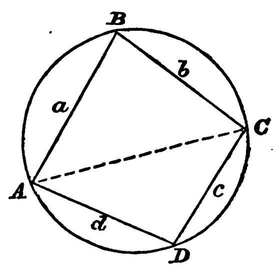
and the formula $$a = (b+c)\cos\theta. \quad [702, 637$$

CASE III.—When the three sides are known, the angles may be found without employing logarithms, from the formula

731 $$\cos A = \frac{b^2+c^2-a^2}{2bc}.$$ [703

732 If logarithms are to be used, take the formulæ for $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, or $\tan\frac{A}{2}$; (704) and (705).

QUADRILATERAL INSCRIBED IN A CIRCLE.



733 $$\cos B = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

From $AC^2 = a^2+b^2-2ab\cos B = c^2+d^2+2cd\cos B$, by (702), and $B+D=180°$.

734 $$\sin B = \frac{2Q}{ab+cd}.$$ [613, 733

735 $$Q = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$ = area of $ABCD$,

and $$s = \tfrac{1}{2}(a+b+c+d).$$

Area $= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin B$; substitute $\sin B$ from last.

736 $$AC^2 = \frac{(ac+bd)(ad+bc)}{(ab+cd)}.$$ [702, 733

Radius of circumscribed circle

737 $$= \frac{1}{4Q}\sqrt{(ab+cd)(ac+bd)(ad+bc)}.$$ [713, 734, 736

If AD bisect the side of the triangle ABC in D,

738 $$\tan BDA = \frac{4\Delta}{b^2-c^2}.$$

739 $$\cot BAD = 2\cot A + \cot B.$$

740 $$AD^2 = \tfrac{1}{4}(b^2+c^2+2bc\cos A) = \tfrac{1}{2}(b^2+c^2-\tfrac{1}{2}a^2).$$

If AD bisect the angle A of a triangle ABC,

742 $$\tan BDA = \cot\frac{B-C}{2} = \frac{b+c}{b-c}\tan\frac{A}{2}.$$

743 $$AD = \frac{2bc}{b+c}\cos\frac{A}{2}.$$

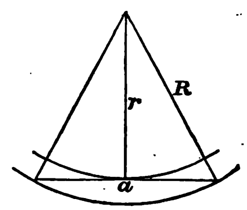
If AD be perpendicular to BC,

744 $$AD = \frac{bc \sin A}{a} = \frac{b^2 \sin C + c^2 \sin B}{b+c}.$$

745 $$BD \sim CD = \frac{b^2-c^2}{a} = a\frac{\tan B - \tan C}{\tan B + \tan C}.$$

REGULAR POLYGON AND CIRCLE.

Radius of circumscribing circle $= R$.
Radius of inscribed circle $= r$.
Side of polygon $= a$.
Number of sides $= n$.



746 $$R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}, \quad r = \frac{a}{2}\cot\frac{\pi}{n}.$$

Area of Polygon

748 $$= \tfrac{1}{4}na^2 \cot\frac{\pi}{n} = \tfrac{1}{2}nR^2 \sin\frac{2\pi}{n} = nr^2 \tan\frac{\pi}{n}.$$

USE OF SUBSIDIARY ANGLES.

749 To adapt $a \pm b$ to logarithmic computation.

Take $\theta = \tan^{-1}\sqrt{\frac{b}{a}}$; then $a+b = a\sec^2\theta$.

750 For $a-b$ take $\theta = \tan^{-1}\left(\frac{b}{a}\right)$; thus

$$a-b = \frac{a\sqrt{2}\cos(\theta+45^\circ)}{\cos\theta}.$$

751 To adapt $a\cos C \pm b\sin C$ to logarithmic computation.

Take $\theta = \tan^{-1}\frac{a}{b}$; then

$$a\cos C \pm b\sin C = \sqrt{(a^2+b^2)}\sin(\theta \pm C).$$ [By 617

For similar instances of the use of a subsidiary angle, see (726) to (730).

752 To solve a quadratic equation by employing a subsidiary angle.

If $x^2 - 2px + q = 0$ be the equation,

$$x = p\left(1 \pm \sqrt{1 - \frac{q}{p^2}}\right).$$ [By 45

CASE I.—If q be $<p^2$, put $\frac{q}{p^2} = \sin^2\theta$; then

$$x = 2p\cos^2\frac{\theta}{2}, \text{ and } 2p\sin^2\frac{\theta}{2}. \qquad [639, 640$$

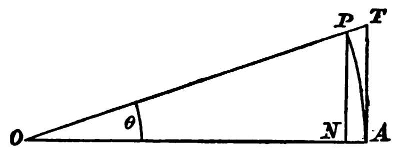
CASE II.—If q be $>p^2$, put $\frac{q}{p^2} = \sec^2\theta$; then

$$x = p\,(1 \pm i\tan\theta), \text{ imaginary roots.} \qquad [614$$

CASE III.—If q be negative, put $\frac{q}{p^2} = \tan^2\theta$; then

$$x = \sqrt{q}\cot\frac{\theta}{2} \text{ and } -\sqrt{q}\tan\frac{\theta}{2}. \qquad [644, 645$$

LIMITS OF RATIOS.



753 $\frac{\sin\theta}{\theta} = \frac{\tan\theta}{\theta} = 1,$

when θ vanishes.

For ultimately $\frac{PN}{AP} = \frac{AT}{AP} = 1$ [601, 606

754 $n\sin\frac{\theta}{n} = \theta$ when n is infinite. By putting $\frac{\theta}{n}$ for θ in last.

755 $\left(\cos\frac{\theta}{n}\right)^n = 1$ when n is infinite.

Put $\left(1-\sin^2\frac{\theta}{n}\right)^{\frac{n}{2}}$, and expand the logarithm by (156).

DE MOIVRE'S THEOREM.

756 $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)\ \ldots$ &c.

$$= \cos(\alpha+\beta+\gamma+\ldots) + i\sin(\alpha+\beta+\gamma+\ldots),$$

where $i = \sqrt{-1}$. Proved by Induction.

757 $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

By Induction, or by putting α, β, &c. each $= \theta$ in (756).

Expansion of $\cos n\theta$, &c. in powers of $\sin\theta$ and $\cos\theta$.

758 $\cos n\theta = \cos^n\theta - C(n,2)\cos^{n-2}\theta\sin^2\theta + C(n,4)\cos^{n-4}\theta\sin^4\theta - \&c.$

759 $\sin n\theta = n\cos^{n-1}\theta\sin\theta - C(n,3)\cos^{n-3}\theta\sin^3\theta + \&c.$

By expanding (757) by Bin. Th., and equating real and imaginary parts.

760 $\tan n\theta = \dfrac{n\tan\theta - C(n,3)\tan^3\theta + \&c.}{1 - C(n,2)\tan^2\theta + C(n,4)\tan^4\theta - \&c.}$.

In series (758, 759), stop at, and exclude, all terms with indices greater than n. Note, *n is here an integer.*

Let s_r = sum of the $C(n, r)$ products of $\tan\alpha$, $\tan\beta$, $\tan\gamma$, &c. to n terms.

761 $\sin(\alpha+\beta+\gamma+\&c.) = \cos\alpha\cos\beta\ldots(s_1 - s_3 + s_5 - \&c.)$

762 $\cos(\alpha+\beta+\gamma+\&c.) = \cos\alpha\cos\beta\ldots(1 - s_2 + s_4 - \&c.)$

By equating real and imaginary parts in (756).

763 $\tan(\alpha+\beta+\gamma+\&c.) = \dfrac{s_1 - s_3 + s_5 - s_7 + \&c.}{1 - s_2 + s_4 - s_6 + \&c.}$.

Expansions of the sine and cosine in powers of the angle.

764 $\sin\theta = \theta - \dfrac{\theta^3}{\lfloor 3} + \dfrac{\theta^5}{\lfloor 5} - \&c.\quad \cos\theta = 1 - \dfrac{\theta^2}{\lfloor 2} + \dfrac{\theta^4}{\lfloor 4} - \&c.$

By putting $\dfrac{\theta}{n}$ for θ in (757) and $n=\infty$, employing (754) and (755).

---

766 $e^{i\theta} = \cos\theta + i\sin\theta.\quad e^{-i\theta} = \cos\theta - i\sin\theta.$ By (150).

768 $e^{i\theta} + e^{-i\theta} = 2\cos\theta.\quad e^{i\theta} - e^{-i\theta} = 2i\sin\theta.$

---

770 $i\tan\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}.\quad \dfrac{1 + i\tan\theta}{1 - i\tan\theta} = e^{2i\theta}.$

Expansion of $\cos^n\theta$ and $\sin^n\theta$ in cosines or sines of multiples of θ.

772 $2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta$
$+ C(n,2)\cos(n-4)\theta + C(n,3)\cos(n-6)\theta + \&c.$

773 When n is even,
$2^{n-1}(-1)^{\frac{1}{2}n}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta$
$+ C(n,2)\cos(n-4)\theta - C(n,3)\cos(n-6)\theta + \&c.$

774 And when n is odd,
$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta$
$+ C(n,2)\sin(n-4)\theta - C(n,3)\sin(n-6)\theta + \&c.$

Observe that in these series the coefficients are those of the Binomial Theorem, with this exception,—*If n be even, the last term must be divided by* 2.

The series are obtained by expanding $(e^{i\theta} \pm e^{i\theta})^n$ by the Binomial Theorem, collecting the equidistant terms in pairs, and employing (768) and (769).

Expansion of cos nθ and sin nθ in powers of sin θ.

775 When n is even,

$$\cos n\theta = 1 - \frac{n^2}{\lfloor 2}\sin^2\theta + \frac{n^2(n^2-2^2)}{\lfloor 4}\sin^4\theta - \frac{n^2(n^2-2^2)(n^2-4^2)}{\lfloor 6}\sin^6\theta + \&c.$$

776 When n is odd,

$$\cos n\theta = \cos\theta\left\{1 - \frac{n^2-1}{\lfloor 2}\sin^2\theta + \frac{(n^2-1)(n^2-3^2)}{\lfloor 4}\sin^4\theta - \frac{(n^2-1)(n^2-3^2)(n^2-5^2)}{\lfloor 6}\sin^6\theta + \&c.\right\}$$

777 When n is even,

$$\sin n\theta = n\cos\theta\left\{\sin\theta - \frac{n^2-2^2}{\lfloor 3}\sin^3\theta + \frac{(n^2-2^2)(n^2-4^2)}{\lfloor 5}\sin^5\theta - \frac{(n^2-2^2)(n^2-4^2)(n^2-6^2)}{\lfloor 7}\sin^7\theta + \&c.\right\}$$

778 When n is odd,

$$\sin n\theta = n\sin\theta - \frac{n(n^2-1)}{\lfloor 3}\sin^3\theta + \frac{n(n^2-1)(n^2-3^2)}{\lfloor 5}\sin^5\theta - \frac{n(n^2-1)(n^2-3^2)(n^2-5^2)}{\lfloor 7}\sin^7\theta + \&c.$$

METHOD OF PROOF.

By (758), we may assume, when n is an even integer,

$$\cos n\theta = 1 + A_2\sin^2\theta + A_4\sin^4\theta + \ldots + A_{2r}\sin^{2r}\theta + \ldots$$

Put $\theta + x$ for θ, and in $\cos n\theta\cos nx - \sin n\theta\sin nx$ substitute for $\cos nx$ and $\sin nx$ their values in powers of nx from (764). Each term on the right is of the type $A_{2r}(\sin\theta\cos x + \cos\theta\sin x)^{2r}$. Make similar substitutions for $\cos x$ and $\sin x$ in powers of x. Collect the two coefficients of x^2 in each term by the multinomial theorem (137) and equate them all to the coefficient of x^2 on the left. In this equation write $\cos^2\theta$ for $1-\sin^2\theta$ everywhere, and then equate the coefficients of $\sin^{2r}\theta$ to obtain the relation between the successive quantities A_{2r} and A_{2r+2} for the series (775).

To obtain the series (777) equate the coefficients of x instead of those of x^2.

When n is an odd integer, begin by assuming, by (759),

$$\sin n\theta = A_1\sin\theta + A_3\sin^3\theta + \&c.$$

779 The expansions of $\cos n\theta$ and $\sin n\theta$ in powers of $\cos\theta$ are obtained by changing θ into $\frac{1}{2}\pi-\theta$ in (775) to (778).

780 *Expansion of* $\cos n\theta$ *in descending powers of* $\cos\theta$.

$$2\cos n\theta=(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\frac{n(n-3)}{\lfloor 2}(2\cos\theta)^{n-4}-$$

$$\ldots+(-1)^r\frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{\lfloor r}(2\cos\theta)^{n-2r}+$$

up to the last positive power of $2\cos\theta$.

Obtained by expanding each term of the identity

$$\log(1-zx)+\log\left(1-\frac{z}{x}\right)=\log\left\{1-z\left(x+\frac{1}{x}-z\right)\right\}$$

by (156), equating coefficients of z^n, and substituting from (768).

783 $\sin\alpha+c\sin(\alpha+\beta)+c^2\sin(\alpha+2\beta)+\&c.$ to n terms

$$=\frac{\sin\alpha-c\sin(\alpha-\beta)-c^n\sin(\alpha+n\beta)+c^{n+1}\sin\{\alpha+\overline{n-1}\,\beta\}}{1-2c\cos\beta+c^2}.$$

If c be <1 and n infinite, this becomes

784 $$=\frac{\sin\alpha-c\sin(\alpha-\beta)}{1-2c\cos\beta+c^2}.$$

785 $\cos\alpha+c\cos(\alpha+\beta)+c\cos(\alpha+2\beta)+\&c.$ to n terms $=$ a similar result, changing *sin* into *cos* in the numerator.

786 Similarly when c is <1 and n infinite.

787 ***Method of summation.***—*Substitute for the sines or cosines their exponential values* (768). *Sum the two resulting geometrical series, and substitute the sines or cosines again for the exponential values by* (766).

788 $c\sin(\alpha+\beta)+\frac{c^2}{\lfloor 2}\sin(\alpha+2\beta)+\frac{c^3}{\lfloor 3}\sin(\alpha+3\beta)+\&c.$ to infinity $=e^{c\cos\beta}\sin(\alpha+c\sin\beta)-\sin\alpha.$

789 $c\cos(\alpha+\beta)+\frac{c^2}{\lfloor 2}\cos(\alpha+2\beta)+\frac{c^3}{\lfloor 3}\cos(\alpha+3\beta)+\&c.$ to infinity $=e^{c\cos\beta}\cos(\alpha+c\sin\beta)-\cos\alpha.$

Obtained by the rule in (787).

790 If, in the series (783) to (789), β be changed into $\beta+\pi$, the signs of the alternate terms will thereby be changed.

Expansion of θ in powers of $\tan\theta$ (Gregory's series).

791 $$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \&c.$$

The series converges if $\tan\theta$ be not > 1.

Obtained by expanding the logarithm of the value of $e^{2i\theta}$ in (771) by (158).

Formulæ for the calculation of the value of π by Gregory's series.

792 $$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$ [791

794 $$= 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}.$$

Proved by employing the formula for $\tan(A \pm B)$, (631).

To prove that π is incommensurable.

795 Convert the value of $\tan\theta$ in terms of θ from (764) and (765) into a continued fraction, thus $\tan\theta = \frac{\theta}{1-}\,\frac{\theta^2}{3-}\,\frac{\theta^2}{5-}\,\frac{\theta^2}{7-}\,\&c.$; or this result may be obtained by putting $i\theta$ for y in (294), and by (770). Hence

$$1 - \frac{\theta}{\tan\theta} = \frac{\theta^2}{3-}\,\frac{\theta^2}{5-}\,\frac{\theta^2}{7-}\,\&c.$$

Put $\frac{\pi}{2}$ for θ, and assume that π, and therefore $\frac{\pi^2}{4}$, *is* commensurable. Let $\frac{\pi^2}{4} = \frac{m}{n}$, m and n being integers. Then we shall have $1 = \frac{m}{3n-}\,\frac{mn}{5n-}\,\frac{mn}{7n-}\,\&c.$ The continued fraction is incommensurable, by (177). But unity cannot be equal to an incommensurable quantity. Therefore π is *not* commensurable.

796 If $\sin x = n\sin(x+a)$, $x = n\sin a + \frac{n^2}{2}\sin 2a + \frac{n^3}{3}\sin 3a + \&c.$

797 If $\tan x = n\tan y$, $x = y - m\sin 2y + \frac{m^2}{2}\sin 4y - \frac{m^3}{3}\sin 6y + \&c.$, where $m = \frac{1-n}{1+n}$.

These results are obtained by substituting the exponential values of the sine or tangent (769) *and* (770), *and then eliminating x.*

798 Coefficient of x^n in the expansion of $e^{ax}\cos bx = \frac{(a^2+b^2)^{\frac{n}{2}}}{\lfloor n}\cos n\theta$, where $a = r\cos\theta$ and $b = r\sin\theta$.

For proof, substitute for $\cos bx$ from (768); expand by (150); put $a = r\cos\theta$, $b = r\sin\theta$ in the coefficient of x^n, and employ (757).

799 When e is < 1, $\dfrac{\sqrt{1-e^2}}{1-e\cos\theta} = 1+2b\cos\theta+2b^2\cos 2\theta+2b^3\cos 3\theta+\ldots$,
where $b = \dfrac{e}{1+\sqrt{1-e^2}}$.

For proof, put $e = \dfrac{2b}{1+b^2}$ and $2\cos\theta = x+\dfrac{1}{x}$, expand the fraction in two series of powers of x by the method of (257), and substitute from (768).

---

800 $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin\{\alpha+(n-1)\beta\}$

$$= \frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n}{2}\beta}{\sin\frac{\beta}{2}}.$$

801 $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos\{\alpha+(n-1)\beta\}$

$$= \frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

802 If the terms in these series have the signs + and − alternately, change β into $\beta+\pi$ in the results.

For proof, multiply the series by $2\sin\frac{\beta}{2}$, and apply (669) and (666).

---

803 If $\beta = \dfrac{2\pi}{n}$ in (800) and (801), each series vanishes.

804 Generally, If $\beta = \dfrac{2\pi}{n}$, and if r be an integer not a multiple of n, the sum of the r^{th} powers of the sines or cosines in (800) or (801) is zero if r be odd; and if r be even it is $= \dfrac{n}{2^r}C\left(r, \dfrac{r}{2}\right)$; by (772) to (774).

General Theorem.—Denoting the sum of the series

805 ...... $c+c_1x+c_2x^2+\ldots+c_nx^n$ by $F(x)$;

then $c\cos\alpha+c_1\cos(\alpha+\beta)+\ldots+c_n\cos(\alpha+n\beta) = \frac{1}{2}\{e^{i\alpha}F(e^{i\beta})+e^{-i\alpha}F(e^{-i\beta})\}$,
and ......

806 $c\sin\alpha+c_1\sin(\alpha+\beta)+\ldots+c_n\sin(\alpha+n\beta) = \dfrac{1}{2i}\{e^{i\alpha}F(e^{i\beta})-e^{-i\alpha}F(e^{-i\beta})\}$.

Proved by substituting for the sines and cosines their exponential values (766), &c.

Expansion of the sine and cosine in factors.

807 $$x^{2n} - 2x^n y^n \cos n\theta + y^{2n}$$

$$= \left\{x^2 - 2xy\cos\theta + y^2\right\}\left\{x^2 - 2xy\cos\left(\theta + \frac{2\pi}{n}\right) + y^2\right\}\dots$$

to n factors, adding $\frac{2\pi}{n}$ to the angle successively.

PROOF.—By solving the quadratic on the left, we get $x = y(\cos n\theta + i\sin n\theta)^{\frac{1}{n}}$. The n values of x are found by (757) and (626), and thence the factors. For the factors of $x^n \pm y^n$ see (480).

---

808 $$\sin n\phi = 2^{n-1}\sin\phi\sin\left(\phi + \frac{\pi}{n}\right)\sin\left(\phi + \frac{2\pi}{n}\right)\dots$$

as far as n factors of sines.

PROOF.—By putting $x = y = 1$ and $\theta = 2\phi$ in the last.

809 If n be even,

$$\sin n\phi = 2^{n-1}\sin\phi\cos\phi\left(\sin^2\frac{\pi}{n} - \sin^2\phi\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\phi\right)\text{\&c.}$$

810 If n be odd, omit $\cos\phi$ and make up n factors, reckoning two factors for each pair of terms in brackets.

Obtained from (808), by collecting equidistant factors in pairs, and applying (659).

---

811 $$\cos n\phi = 2^{n-1}\sin\left(\phi + \frac{\pi}{2n}\right)\sin\left(\phi + \frac{3\pi}{2n}\right)\dots \text{ to } n \text{ factors.}$$

PROOF.—Put $\phi + \frac{\pi}{2n}$ for ϕ in (808).

812 Also, if n be odd,

$$\cos n\phi = 2^{n-1}\cos\phi\left(\sin^2\frac{\pi}{2n} - \sin^2\phi\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\phi\right)\dots$$

813 If n be even, omit $\cos\phi$.

Proved as in (809).

---

814 $$n = 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\dots\sin\frac{(n-1)\pi}{n}.$$

PROOF.—Divide (809) by $\sin\phi$, and make ϕ vanish; then apply (754).

---

815 $$\sin\theta = \theta\left\{1 - \left(\frac{\theta}{\pi}\right)^2\right\}\left\{1 - \left(\frac{\theta}{2\pi}\right)^2\right\}\left\{1 - \left(\frac{\theta}{3\pi}\right)^2\right\}\dots\dots$$

816 $$\cos\theta = \left\{1 - \left(\frac{2\theta}{\pi}\right)^2\right\}\left\{1 - \left(\frac{2\theta}{3\pi}\right)^2\right\}\left\{1 - \left(\frac{2\theta}{5\pi}\right)^2\right\}\dots\dots$$

PROOF.—Put $\phi = \frac{\theta}{n}$ in (809) and (812); divide by (814) and make n infinite.

817 $e^{x}-2\cos\theta+e^{-x}$

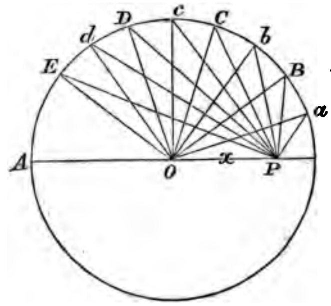
$$=4\sin^2\frac{\theta}{2}\left\{1+\frac{x^2}{\theta^2}\right\}\left\{1+\frac{x^2}{(2\pi\pm\theta)^2}\right\}\left\{1+\frac{x^2}{(4\pi\pm\theta)^2}\right\}\dots$$

Proved by substituting $x=1+\frac{z}{2n}$, $y=1-\frac{z}{2n}$, and $\frac{\theta}{n}$ for θ in (807), making n infinite and reducing one series of factors to $4\sin^2\frac{\theta}{2}$ by putting $z=0$.

De Moivre's Property of the Circle. — Take P any point, and $POB=\theta$ any angle,

$$BOC=COD=\&c.=\frac{2\pi}{n};$$

$$OP=x;\quad OB=r.$$



819 $x^{2n}-2x^nr^n\cos n\theta+r^{2n}$
$=PB^2\,PC^2\,PD^2\dots$ to n factors.

By (807) and (702), since $PB^2=x^2-2xr\cos\theta+r^2$, &c.

820 If $x=r$, $2r^n\sin\frac{n\theta}{2}=PB.PC.PD\dots$ &c.

821 *Cotes's properties.*—If $\theta=\frac{2\pi}{n}$,

$$x^n\sim r^n=PB.PC.PD\dots\ \&c.$$

822 $$x^n+r^n=Pa.Pb.Pc\dots\ \&c.$$

ADDITIONAL FORMULÆ.

823 $\cot A+\tan A=2\operatorname{cosec}2A=\sec A\operatorname{cosec}A$.

824 $\operatorname{cosec}2A+\cot 2A=\cot A$. $\sec A=1+\tan A\tan\frac{A}{2}$.

826 $$\cos A=\cos^4\frac{A}{2}-\sin^4\frac{A}{2}.$$

827 $$\tan A+\sec A=\tan\left(45^\circ+\frac{A}{2}\right).$$

828 $$\frac{\tan A+\tan B}{\cot A+\cot B}=\tan A\tan B.$$

829. $$\sec^2 A\operatorname{cosec}^2 A=\sec^2 A+\operatorname{cosec}^2 A.$$

830 If $A+B+C=\frac{\pi}{2}$,

$$\tan B\tan C+\tan C\tan A+\tan A\tan B=1.$$

831 If $A+B+C=\pi$,

$$\cot B\cot C+\cot C\cot A+\cot A\cot B=1.$$

---

832 $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}=\frac{\pi}{2}.\quad \tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}.$

In a right-angled triangle ABC, C being the right angle,

833 $\cos 2B=\frac{a^2-b^2}{a^2+b^2}.\qquad \tan 2B=\frac{2ab}{a^2-b^2}.$

834 $\tan\frac{1}{2}A=\sqrt{\left(\frac{c-b}{c+b}\right)}.\qquad R+r=\frac{1}{2}(a+b).$

In any triangle,

835 $\sin\frac{1}{2}(A-B)=\frac{a-b}{c}\cos\frac{1}{2}C.$

$$\cos\frac{1}{2}(A-B)=\frac{a+b}{c}\sin\frac{1}{2}C.$$

836 $\frac{\sin A-B}{\sin A+B}=\frac{a^2-b^2}{c^2}.\qquad \frac{\tan\frac{1}{2}A+\tan\frac{1}{2}B}{\tan\frac{1}{2}A-\tan\frac{1}{2}B}=\frac{c}{a-b}.$

837 $\frac{1}{2}(a^2+b^2+c^2)=bc\cos A+ca\cos B+ab\cos C.$

---

838 Area of triangle $ABC=\frac{1}{2}bc\sin A$

$$=\frac{1}{2}a^2\frac{\sin B\sin C}{\sin A}=\frac{1}{2}(a^2-b^2)\frac{\sin A\sin B}{\sin(A-B)}.$$

839 $=\frac{2abc}{a+b+c}\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C.$

840 $=\frac{1}{4}(a+b+c)^2\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C.$

With the notation of (709),

841 $r=\frac{1}{2}(a+b+c)\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C.$

842 $2Rr=\frac{abc}{a+b+c}.\qquad \Delta=\sqrt{rr_ar_br_c}.$

843 $a\cos A+b\cos B+c\cos C=4R\sin A\sin B\sin C.$

---

844 $R+r = \frac{1}{2}(a\cot A + b\cot B + c\cot C)$ = sum of perpendiculars on the sides from centre of circumscribing circle.

This may also be shown by applying Euc. VI. D. to the circle described on R as diameter and the quadrilateral so formed.

845 $$r_a r_b r_c = abc \cos\tfrac{1}{2}A \cos\tfrac{1}{2}B \cos\tfrac{1}{2}C.$$

846 $$r = \sqrt{(r_b r_c)} + \sqrt{(r_c r_a)} + \sqrt{(r_a r_b)}.$$

847 $$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}. \qquad \tan\tfrac{1}{2}A = \sqrt{\frac{r r_a}{r_b r_c}}.$$

849 If O be the centre of inscribed circle,

$$OA = \frac{2bc}{a+b+c}\cos\tfrac{1}{2}A.$$

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850 $$a(b\cos C - c\cos B) = b^2 - c^2.$$

851 $$b\cos B + c\cos C = c\cos(B-C).$$

852 $$a\cos A + b\cos B + c\cos C = 2a\sin B\sin C.$$

853 $$\cos A + \cos B + \cos C = 1 + \frac{2a\sin B\sin C}{a+b+c}.$$

---

854 If $s = \frac{1}{2}(a+b+c)$,

$$1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c$$
$$= 4\sin s\sin(s-a)\sin(s-b)\sin(s-c).$$

855 $$-1 + \cos^2 a + \cos^2 b + \cos^2 c + 2\cos a\cos b\cos c$$
$$= 4\cos s\cos(s-a)\cos(s-b)\cos(s-c).$$

856 $$4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}$$
$$= \cos s + \cos(s-a) + \cos(s-b) + \cos(s-c).$$

857 $$4\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}$$
$$= -\sin s + \sin(s-a) + \sin(s-b) + \sin(s-c).$$

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858 $$\pi^2 = 6\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) = 8\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right).$$

* Proved by equating coefficients of $\frac{\sin\theta}{\theta}$ in (764 and 815) or of $\cos\theta$ in (765 and 816).

859 *Examples of the Solution of Triangles.*

Ex. 1: Case II. (724).—Two sides of a triangle b, c, being 900 and 700 feet, and the included angle 47° 25′, to find the remaining angles.

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{1}{8}\cot 23°\,42'\,30'';$$

therefore $\log\tan\tfrac{1}{2}(B-C) = \log\cot\frac{A}{2} - \log 8;$

therefore $L\tan\tfrac{1}{2}(B-C) = L\cot 23°\,42'\,30'' - 3\log 2;$

10 being added to each side of the equation.

$\therefore\ L\cot 23°\,42'\,30'' = 10{\cdot}3573942$ *
$\quad 3\log 2 = {\cdot}9030900$
$\therefore\ L\tan\tfrac{1}{2}(B-C) = 9{\cdot}4543042$

$\therefore\ \tfrac{1}{2}(B-C) = 15°\,53'\,19''{\cdot}55$ *
and $\tfrac{1}{2}(B+C) = 66°\,17'\,30''$
$\therefore\ B = 82°\,10'\,49''{\cdot}55$

And, by subtraction, $C = 50°\,24'\,10''{\cdot}45$

Ex. 2: Case III. (732).—Given the sides $a, b, c = 7, 8, 9$ respectively, to find the angles.

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{4.3}{12.5}} = \sqrt{\frac{2}{10}},$$

therefore $L\tan\frac{A}{2} = 10 + \tfrac{1}{2}(\log 2 - 1) = 9{\cdot}650515;$

therefore $\tfrac{1}{2}A = 24°\,5'\,41''{\cdot}43.$*

$\tfrac{1}{2}B$ is found in a similar manner, and $C = 180° - A - B$.

Ex. 3.—In a right-angled triangle, given the hypotenuse $c = 6953$ and a side $b = 3$, to find the remaining angle.

Here $\cos A = \frac{3}{6953}$. But, since A is nearly a right angle, it cannot be determined accurately from $\log\cos A$. Therefore take

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{3475}{6953}},$$

therefore $L\sin\frac{A}{2} = 10 + \tfrac{1}{2}(\log 3475 - \log 6953) = 9{\cdot}8493913,$

therefore $\frac{A}{2} = 44°\,59'\,15''{\cdot}52,$*

therefore $A = 89°\,58'\,31''{\cdot}04$ and $B = 0°\,1'\,28''{\cdot}96.$

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* See Chambers's Mathematical Tables for a concise explanation of the method of obtaining these figures.

SPHERICAL TRIGONOMETRY.

INTRODUCTORY THEOREMS.

870 *Definitions.*—Planes through the centre of a sphere intersect the surface in *great circles;* other planes intersect it in *small circles.* Unless otherwise stated, all arcs are measured on great circles.

The *poles* of a great circle are the extremities of the diameter perpendicular to its plane.

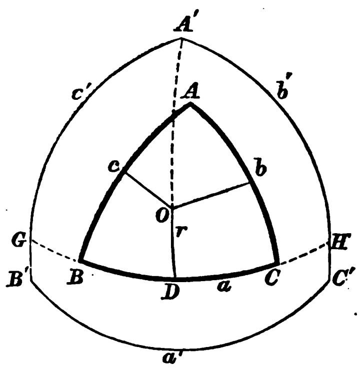
The sides a, b, c of a spherical triangle are the arcs of great circles BC, CA, AB on a sphere of radius unity; and the angles A, B, C are the angles between the tangents to the sides at the vertices, or the angles between the planes of the great circles. The centre of the sphere will be denoted by O.

The *polar triangle* of a spherical triangle ABC has for its angular points A', B', C', the poles of the sides BC, CA, AB of the *primitive triangle* in the directions of A, B, C respectively (since each great circle has two poles). The sides of $A'B'C'$ are denoted by a', b', c'.

871 The sides and angles of the polar triangle are respectively the supplements of the angles and sides of the primitive triangle; that is,

$$a'+A = b'+B = c'+C = \pi,$$

$$a+A' = b+B' = c+C' = \pi.$$



Let BC produced cut the sides $A'B'$, $C'A'$ in G, H. B is the pole of $A'C'$, therefore $BH = \frac{\pi}{2}$. Similarly $CG = \frac{\pi}{2}$, therefore, by addition, $a+GH = \pi$ and $GH = A'$, because A' is the pole of BC.

The polar diagram of a spherical polygon is formed in the same way, and the same relations subsist between the sides and angles of the two figures.

2 B

RULE.—*Hence, any equation between the sides and angles of a spherical triangle produces a supplementary equation by changing* a *into* $\pi - A$ *and* A *into* $\pi - a$, *&c.*

872 The centre of the inscribed circle, radius r, is also the centre of the circumscribed circle, radius R', of the polar triangle, and $r + R' = \frac{1}{2}\pi$.

PROOF.—In the last figure, let O be the centre of the inscribed circle of ABC; then OD, the perpendicular on BC, passes through A', the pole of BC. Also, $OD = r$, therefore $OA' = \frac{1}{2}\pi - r$. Similarly $OB' = OC' = \frac{1}{2}\pi - r$, therefore O is the centre of the circumscribed circle of $A'B'C'$, and $r + R' = \frac{1}{2}\pi$.

873 The sine of the arc joining a point on the circumference of a small circle with the pole of a parallel great circle is equal to the ratio of the circumferences, or corresponding arcs of the two circles.

For it is equal to the radius of the small circle divided by the radius of the sphere; that is, by the radius of the great circle.

874 Two sides of a triangle are greater than the third. [By XI. 20.

875 The sides of a triangle are together less than the circumference of a great circle. [By XI. 21.

876 The angles of a triangle are together greater than two right angles.

For $\pi - A + \pi - B + \pi - C$ is $< 2\pi$, by (875) and the polar triangle.

877 If two sides of a triangle are equal, the opposite angles are equal. [By the geometrical proof in (894).

878 If two angles of a triangle are equal, the opposite sides are equal. [By the polar triangle and (877).

879 The greater angle of a triangle has the greater side opposite to it.

PROOF.—If B be $>A$, draw the arc BD meeting AC in D, and make $\angle ABD = A$, therefore $BD = AD$; but $BD + DC > BC$, therefore $AC > BC$.

880 The greater side of a triangle has the greater angle opposite to it. [By the polar triangle and (879).

RIGHT-ANGLED TRIANGLES.

881 ***Napier's Rules.***—In the triangle ABC let C be a right angle, then a, $(\frac{1}{2}\pi - B)$, $(\frac{1}{2}\pi - c)$, $(\frac{1}{2}\pi - A)$, and b, are called the five *circular parts.* Taking any part for middle part, Napier's rules are—

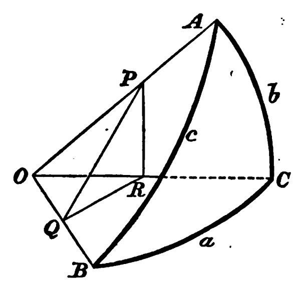
I. *sine of middle part = product of tangents of adjacent parts.*

II. *sine of middle part = product of cosines of opposite parts.*

In applying the rules we can take A, B, c instead of their complements, and change sine into cos, or *vice versâ*, for those parts at once. Thus, taking b for the middle part,

$$\sin b = \tan a \cot A = \sin B \sin c.$$

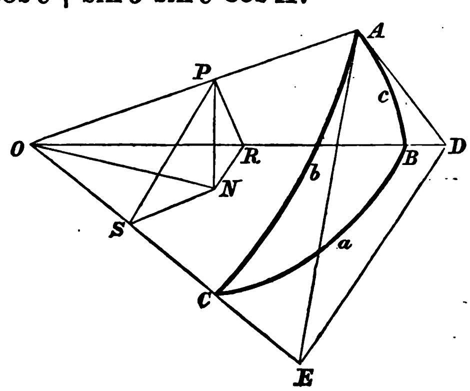
Ten equations in all are given by the rules.



PROOF.—From any point P in OA, draw PR perpendicular to OC, and RQ to OB; therefore PRQ is a right angle, therefore OB is perpendicular to PR and QR, and therefore to PQ. Then prove any formula by proportion from the triangles of the tetrahedron $OPQR$, which are all right-angled. Otherwise, prove by the formulæ for oblique-angled triangles.

OBLIQUE-ANGLED TRIANGLES.

882 $$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$



PROOF.—Draw tangents at A to the sides c, b to meet OB, OC in D and E. Express DE^2 by (702) applied to each of the triangles DAE and DOE, and subtract.

If AB and AC are both $> \frac{\pi}{2}$, produce them to meet in A' the pole of A, and employ the triangle $A'BC$.

If AB alone be $> \frac{\pi}{2}$, produce BA to meet BC.

The supplementary formula, by (871), is

883 $$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

884 $$\sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}.$$

885 $$\cos\frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$$

886 $$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}} \text{ where } s = \tfrac{1}{2}(a+b+c).$$

PROOF.—$\sin^2\frac{A}{2} = \frac{1}{2}(1-\cos A)$. Substitute for $\cos A$ from (882), and throw the numerator of the whole expression into factors by (673). Similarly for $\cos\frac{A}{2}$.

The supplementary formulæ are obtained in a similar way, or by the rule in (871). They are

887 $$\cos\frac{a}{2} = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}.$$

888 $$\sin\frac{a}{2} = \sqrt{\frac{-\cos S\cos(S-A)}{\sin B \sin C}}.$$

889 $$\tan\frac{a}{2} = \sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}$$

where $S = \frac{1}{2}(A+B+C)$.

---

890 Let $\sigma = \sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}$

$$= \tfrac{1}{2}\sqrt{1+2\cos a\cos b\cos c-\cos^2 a-\cos^2 b-\cos^2 c}.$$

Then the supplementary form, by (871), is

891 $\Sigma = \sqrt{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)}$

$$= \tfrac{1}{2}\sqrt{1-2\cos A\cos B\cos C-\cos^2 A-\cos^2 B-\cos^2 C}.$$

---

892 $$\sin A = \frac{2\sigma}{\sin b \sin c}. \qquad \sin a = \frac{2\Sigma}{\sin B \sin C}.$$

[By $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$ and (884, 885), &c.

---

893 The following rules will produce the ten formulæ (884 to 892)—

I. *Write* sin *before each factor in the* s *values of* $\sin\frac{A}{2}$,

$\cos\frac{A}{2}$, $\tan\frac{A}{2}$, $\sin A$, *and* Δ, *in Plane Trigonometry* (704–707), *to obtain the corresponding formulæ in Spherical Trigonometry.*

II. *To obtain the supplementary forms of the five results, transpose large and small letters everywhere, and transpose* sin *and* cos *everywhere but in the denominators, and write minus before* cos S.

894 $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

PROOF.—By (882). Otherwise, in the figure of 882, draw PN perpendicular to BOC, and NR, NS to OB, OC. Prove PRO and PSO right angles by I. 47, and therefore $PN = OP \sin c \sin B = OP \sin b \sin C$.

895 $$\cos b \cos C = \cot a \sin b - \cot A \sin C.$$

To remember this formula, take any four consecutive angles and sides (as a, C, b, A), and calling the first and fourth the extremes, and the second and third the middle parts, employ the following rule:—

RULE.—*Product of cosines of middle parts = cot extreme side × sin middle side − cot extreme angle × sin middle angle.*

PROOF.—In the formula for $\cos a$ (882) substitute a similar value for $\cos c$, and for $\sin c$ put $\sin C \frac{\sin a}{\sin A}$.

896 *NAPIER'S FORMULÆ.*

(1) $$\tan\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

(2) $$\tan\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

(3) $$\tan\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

(4) $$\tan\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

RULE.—*In the value of* $\tan\frac{1}{2}(A-B)$ *change* sin *to* cos *to obtain* $\tan\frac{1}{2}(A+B)$. *To obtain* (3) *and* (4) *from* (1) *and* (2), *transpose sides and angles, and change* cot *to* tan.

PROOF.—In the values of $\cos A$ and $\cos B$, by (883), put $m \sin a$ and $m \sin b$ for $\sin A$ and $\sin B$, and add the two equations. Then put $m = \frac{\sin A \pm \sin B}{\sin a \pm \sin b}$, and transform by (670–672).

897 *GAUSS'S FORMULÆ.*

(1) $$\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}C} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}.$$

(2) $$\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C} = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}.$$

(3) $$\frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}C} = \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}.$$

(4) $$\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C} = \frac{\sin\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}.$$

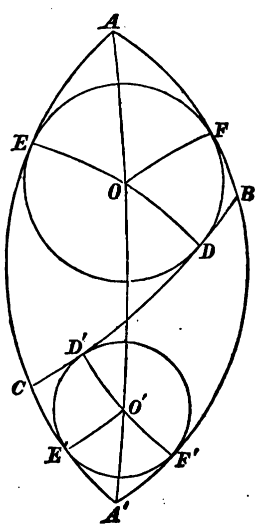
From any one of these formulæ the others may be obtained by the following rule:—

Rule.—*Change the sign of the letter* B *(large or small) on one side of the equation, and write* sin *for* cos *and* cos *for* sin *on the other side.*

Proof.—Take $\sin\frac{1}{2}(A+B) = \sin\frac{1}{2}A\cos\frac{1}{2}B + \cos\frac{1}{2}A\sin\frac{1}{2}B$, substitute the s values by (884, 885), and reduce.

SPHERICAL TRIANGLE AND CIRCLE.

898 Let r be the radius of the inscribed circle of ABC; r_a the radius of the escribed circle touching the side a, and R, R_a the radii of the circumscribed circles; then



(1) $$\tan r = \tan\tfrac{1}{2}A\sin(s-a) = \frac{\sigma}{\sin s}$$

(3) $$= \frac{2\sin a}{\sin A}\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$$

(4) $$= \frac{\Sigma}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$= \frac{2\Sigma}{\cos S+\cos(S-A)+\cos(S-B)+\&c.}$$

Proof. — The first value is found from the right-angled triangle OAF, in which $AF = s-a$. The other values by (884–892).

899 (1) $\tan r_a = \tan \frac{1}{2}A \sin s = \dfrac{\sigma}{\sin(s-a)}$

(3) $= \dfrac{2\sin a}{\sin A} \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$

(4) $= \dfrac{\Sigma}{2\cos \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C}$

$$= \frac{2\Sigma}{-\cos S - \cos(S-A) + \cos(S-B) + \cos(S-C)}.$$

PROOF.—From the right-angled triangle $O'AF'$, in which $AF' = s$.

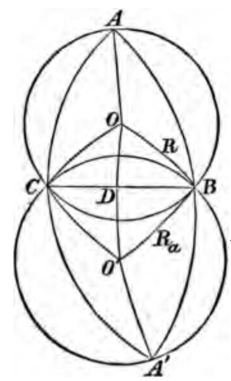
NOTE.—The first two values of $\tan r_a$ may be obtained from those of $\tan r$ by interchanging s and $s-a$.

900 (1) $\tan R = \dfrac{\tan \frac{1}{2}a}{\cos(S-A)} = \dfrac{-\cos S}{\Sigma}$

(3) $= \dfrac{\sin \frac{1}{2}a}{\sin A \cos \frac{1}{2}b \cos \frac{1}{2}c}$

(4) $= \dfrac{2\sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{\sigma}$

$$= \frac{-\sin s + \sin(s-a) + \sin(s-b) + \&c.}{2\sigma}$$



PROOF.—The first value from the right-angled triangle OBD, in which $\angle OBD = S-A$. The other values by the formulæ (887–892).

901 (1) $\tan R_a = \dfrac{\tan \frac{1}{2}a}{-\cos S} = \dfrac{\cos(S-A)}{\Sigma}$

(3) $= \dfrac{\sin \frac{1}{2}a}{\sin A \sin \frac{1}{2}b \sin \frac{1}{2}c}$

(4) $= \dfrac{2\sin \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}{\sigma}$

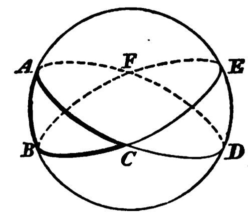
(5) $= \dfrac{\sin s - \sin(s-a) + \sin(s-b) + \sin(s-c)}{2\sigma}.$

PROOF.—From the right-angled triangle $O'BD$, in which $\angle O'BD = \pi - S$.

SPHERICAL AREAS.

902 area of $ABC = (A+B+C-\pi)\,r^2 = Er^2$,

where $E = A+B+C-\pi$, the *spherical excess*.



PROOF.—By adding the three lunes

ABDC, BCEA, CAFB,

and observing that $ABF = CDE$,

we get $\left(\frac{A}{\pi}+\frac{B}{\pi}+\frac{C}{\pi}\right)2\pi r^2 = 2\pi r^2 + 2ABC$.

903 *AREA OF SPHERICAL POLYGON.*

n being the number of sides,

$$\begin{aligned}\text{Area} &= \{\text{Interior Angles} - (n-2)\,\pi\}\,r^2 \\ &= \{2\pi - \text{Exterior Angles}\}\,r^2 \\ &= \{2\pi - \text{sides of Polar Diagram}\}\,r^2.\end{aligned}$$

The last value holds for a curvilinear area in the limit.

PROOF.—By joining the vertices with an interior point, and adding the areas of the spherical triangles so formed.

904 *Cagnoli's Theorem.*

$$\sin\tfrac{1}{2}E = \frac{\sqrt{\{\sin s\,\sin(s-a)\,\sin(s-b)\,\sin(s-c)\}}}{2\cos\frac{1}{2}a\,\cos\frac{1}{2}b\,\cos\frac{1}{2}c}.$$

PROOF.—Expand $\sin[\frac{1}{2}(A+B)-\frac{1}{2}(\pi-C)]$ by (628), and transform by Gauss's equations (897 i., iii.) and (669, 890).

905 *Lhuillier's Theorem.*

$$\tan\tfrac{1}{4}E = \sqrt{\{\tan\tfrac{1}{2}s\,\tan\tfrac{1}{2}(s-a)\,\tan\tfrac{1}{2}(s-b)\,\tan\tfrac{1}{2}(s-c)\}}.$$

PROOF.—Multiply numerator and denominator of the left side by $2\cos\frac{1}{2}(A+B-C+\pi)$ and reduce by (667, 668), then eliminate $\frac{1}{2}(A+B)$ by Gauss's formulæ (897 i., iii.) Transform by (672, 673), and substitute from (886).

POLYHEDRONS.

Let the number of faces, solid angles, and edges, of any polyhedron be F, S, and E; then

906 $$F+S = E+2.$$

Proof.—Project the polyhedron upon an internal sphere. Let m = number of sides, and s = sum of angles of one of the spherical polygons so formed. Then its area $= \{s-(m-2)\pi\} r^2$, by (903). Sum this for all the polygons, and equate to $4\pi r^2$.

THE FIVE REGULAR SOLIDS.

Let m be the number of sides in each face, n the number of plane angles in each solid angle; therefore

907 $$mF = nS = 2E.$$

From these equations and (906), find F, S, and E in terms of m and n, thus,

$$\frac{1}{F} = \frac{m}{2}\left(\frac{1}{m}+\frac{1}{n}-\frac{1}{2}\right), \quad \frac{1}{S} = \frac{n}{2}\left(\frac{1}{m}+\frac{1}{n}-\frac{1}{2}\right), \quad \frac{1}{E} = \frac{1}{m}+\frac{1}{n}-\frac{1}{2}.$$

In order that F, S, and E may be positive, we must have $\frac{1}{m}+\frac{1}{n} > \frac{1}{2}$, a relation which admits of five solutions in whole nnmbers corresponding to the five regular solids. The values of m, n, F, S, and E for the five regular solids are exhibited in the following table:—

	m	n	F	S	E
Tetrahedron	3	3	4	4	6
Hexahedron	4	3	6	8	12
Octahedron	3	4	8	6	12
Dodecahedron	5	3	12	20	30
Icosahedron	3	5	20	12	30

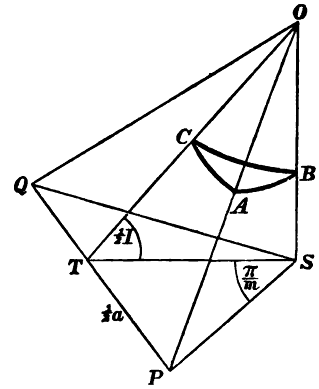
908 The sum of all the plane angles of any polyhedron

$$= 2\pi(S-2);$$

Or, *Four right angles for every vertex less eight right angles.*

909 If I be the angle between two adjacent faces of a regular polyhedron,

$$\sin\tfrac{1}{2}I = \cos\frac{\pi}{n} \div \sin\frac{\pi}{m}.$$



PROOF.—Let $PQ = a$ be the edge, and S the centre of a face, T the middle point of PQ, O the centre of the inscribed and circumscribed spheres, ABC the projection of PST upon a concentric sphere. In this spherical triangle,

$$C = \frac{\pi}{2},\quad A = \frac{\pi}{n},\quad \text{and}\quad B = \frac{\pi}{m} = PST.$$

Also $\qquad STO = \tfrac{1}{2}I.$

Now, by (881, ii.),

$$\cos A = \sin B \cos BC;$$

that is, $\qquad \cos\frac{\pi}{n} = \sin\frac{\pi}{m}\sin\tfrac{1}{2}I.$

Q. e. d.

If r, R be the radii of the inscribed and circumscribed spheres of a regular polyhedron,

910 $\qquad r = \frac{a}{2}\tan\tfrac{1}{2}I\cot\frac{\pi}{m},\quad R = \frac{a}{2}\tan\tfrac{1}{2}I\tan\frac{\pi}{n}.$

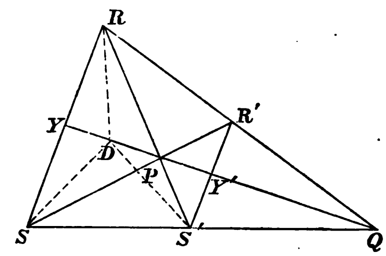
PROOF. — In the above figure, $OS = r$, $OP = R$, $PT = \frac{a}{2}$; and $OS = PT\cot\frac{\pi}{m}\tan\tfrac{1}{2}I$. Also $OP = PT\,\text{cosec}\,AC$, and by (881 i.),

$$\sin AC = \tan BC\cot A = \cot\tfrac{1}{2}I\cot\frac{\pi}{n},\quad \text{therefore, \&c.}$$

ELEMENTARY GEOMETRY.

MISCELLANEOUS PROPOSITIONS.

920 To find the point in a given line QY, the sum of whose distances from two fixed points S, S' is a minimum.



Draw SYR at right angles to QY, making $YR = YS$. Join RS', cutting QY in P. Then P will be the required point.

PROOF.—For, if D be any other point on the line, $SD = DR$ and $SP = PR$. But $RD+DS'$ is $>RS'$, therefore, &c. R is called the reflection of the point S, and SPS' is the path of a ray of light reflected at the line QY.

If S, S' and QY are not in the same plane, make SY, YR equal perpendiculars as before, but the last in the plane of S' and QY.

Similarly, the point Q in the given line, the difference of whose distances from the fixed points S and R' is a maximum, is found by a like construction.

The minimum sum of distances from S, S' is given by

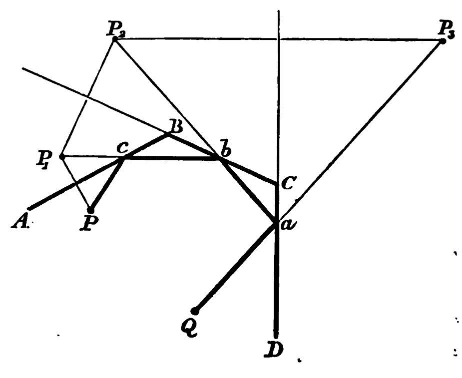
$$(SP+S'P)^2 = SS'^2+4SY.S'Y'.$$

And the maximum difference from S and R' is given by

$$(SQ-R'Q)^2 = (SR')^2-4SY.R'Y'.$$

Proved by VI. D., since $SRR'S'$ can be inscribed in a circle.

921 Hence, to find the shortest distance from P to Q *en route* of the lines AB, BC, CD; in other words, the path of the ray reflected at the successive surfaces AB, BC, CD.



Find P_1, the reflection of P at the first surface; then P_2, the reflection of P_1 at the second surface; next P_3, the reflection of P_2 at the third surface; and so on if there be more surfaces. Lastly, join Q with P_3, the last reflection, cutting CD in a. Join aP_2, cutting BC in b. Join bP_1, cutting AB in c. Join cP. $PcbaQ$ is the path required.

The same construction will give the path when the surfaces are not, as in the case considered, all perpendicular to the same plane.

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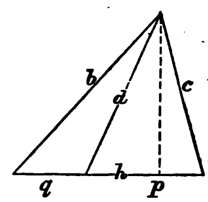
922 If the straight line d from the vertex of a triangle divide the base into segments p, q, and if h be the distance from the point of section to the foot of the perpendicular from the vertex on the base, then

$$b^2+c^2 = p^2+q^2+2d^2+2h(p-q). \qquad \text{[II. 12, 13.}$$

The following cases are important:—

(i.) When $p = q$, $b^2+c^2 = 2q^2+2d^2$;



i.e., the sum of the squares of two sides of a triangle is equal to twice the square of half the base, together with twice the square of the bisecting line drawn from the vertex.

(ii.) When $p = 2q$, $b^2+2c^2 = 6q^2+3d^2$. (II. 12 or 13)

(iii.) When the triangle is isosceles,

$$b^2 = c^2 = pq+d^2.$$

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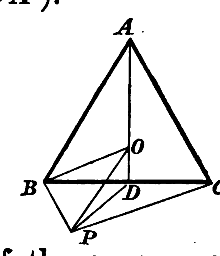
923 If O be the centre of an equilateral triangle ABC and P any point in space. Then

$$PA^2+PB^2+PC^2 = 3(PO^2+OA^2).$$



Proof.— $PB^2+PC^2 = 2PD^2+2BD^2$. (922, i.)

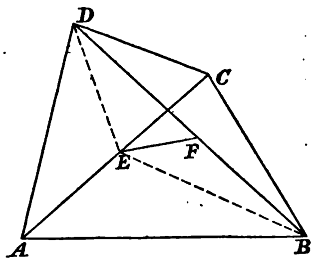
Also $PA^2+2PD^2 = 6OD^2+3PO^2$, (922, ii.)

and $BO = 2OD$;

therefore &c.

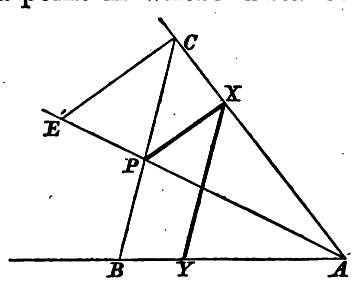
Cor.—Hence, if P be any point on the surface of a sphere, centre O, the sum of the squares of its distances from A, B, C is constant. And if r, the radius of the sphere, be equal to OA, the sum of the same squares is equal to $6r^2$.

924 The sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the line joining the middle points of the diagonals. (922, i.)



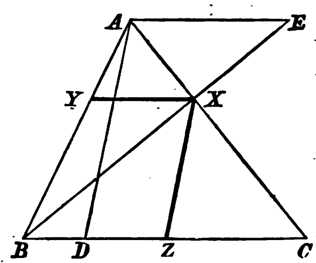
925 Cor.—The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

926 In a given line AC, to find a point X whose distance from a point P shall have a given ratio to its distance in a given direction from a line AB.



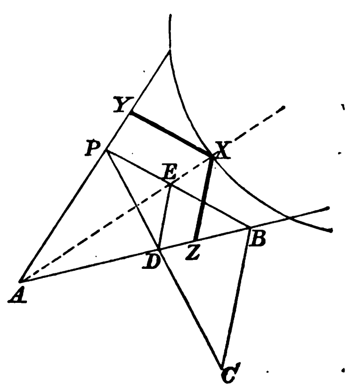
Through P draw BPC parallel to the given direction. Produce AP, and make CE in the given ratio to CB. Draw PX parallel to EC, and XY to CB. There are two solutions when CE cuts AP in two points. [Proof.—By (VI. 2).

927 To find a point X in AC, whose distance XY from AB parallel to BC shall have a given ratio to its distance XZ from BC parallel to AD.



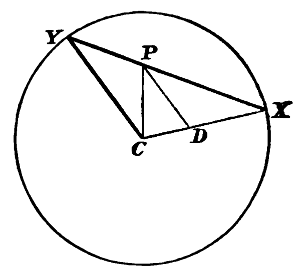
Draw AE parallel to BC, and having to AD the given ratio. Join BE cutting AC in X, the point required. [Proved by (VI. 2).

928 To find a point X on any line, straight or curved, whose distances XY, XZ, in given directions from two given lines AP, AB, shall be in a given ratio.



Take P any point in the first line. Draw PB parallel to the direction of XY, and BC parallel to that of XZ, making PB have to BC the given ratio. Join PC, cutting AB in D. Draw DE parallel to CB. Then AE produced cuts the line in X, the point required, and is the locus of such points. [Proof.—By (VI. 2).

929 To draw a line XY through a given point P so that the segments XP, PY, intercepted by a given circle, shall be in a given ratio.



Divide the radius of the circle in that ratio, and, with the parts for sides, construct a triangle PDC upon PC as base. Produce CD to cut the circle in X. Draw XPY and CY.

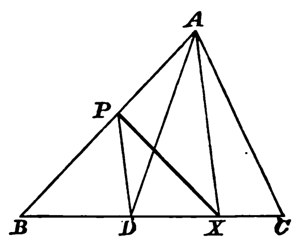
Then $\quad PD+DC =$ radius,

therefore $\quad PD = DX$.

But $\quad CY = CX$,

therefore PD is parallel to CY (I. 5, 28), therefore &c., by (VI. 2).

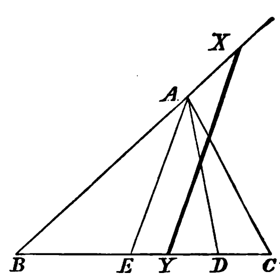
930 From a given point P in the side of a triangle, to draw a line PX which shall divide the area of the triangle in a given ratio.



Divide BC in D in the given ratio, and draw AX parallel to PD. PX will be the line required.

$ABD : ADC =$ the given ratio (VI. 1), and $APD = XPD$ (I. 37), therefore &c.

931 To divide the triangle ABC in a given ratio by a line XY drawn parallel to any given line AE.



Make BD to DC in the given ratio. Then make BY a mean proportional to BE and BD, and draw YX parallel to EA.

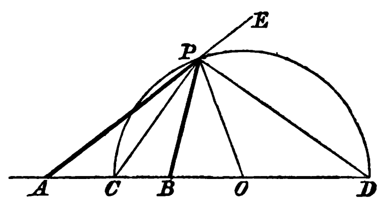
PROOF.—AD divides ABC in the given ratio (VI. 1). Now

$ABE : XBY :: BE : BD$, (VI. 19)

or $\quad :: ABE : ABD$;

therefore $\quad XBY = ABD$.

932 If the interior and exterior vertical angles at P of the triangle APB be bisected by straight lines which cut the base in C and D, then the circle circumscribing CPD gives the locus of the vertices of all triangles on the base AB whose sides AP, PB are in a constant ratio.



PROOF.—
The $\angle CPD = \frac{1}{2}(APB+BPE)$
$= \text{a right angle};$
therefore P lies on the circumference of the circle, diameter CD (III. 31). Also
$AP : PB :: AC : CB :: AD : DB$
(VI. 3, and A.), a fixed ratio.

933 AD is divided harmonically in B and C; *i.e.*, $AD : DB :: AC : CB$; or, *the whole line is to one extreme part as the other extreme part is to the middle part.* If we put a, b, c for the lengths AD, BD, CD, the proportion is expressed algebraically by $a : b :: a-c : c-b$, which is equivalent to

$$\frac{1}{a}+\frac{1}{b}=\frac{2}{c}.$$

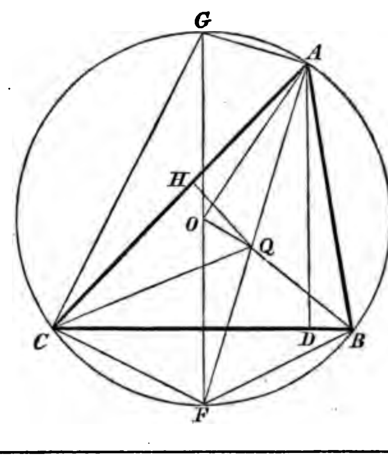
934 Also $AP : BP = OA : OC = OC : OB$
and $AP^2 : BP^2 = OA : OB$, (VI. 19)
$AP^2-AC^2 : CP^2 : BP^2-BC^2$. (VI. 3, & B.)

---

935 If Q be the centre of the inscribed circle of the triangle ABC, and if AQ produced meet the circumscribed circle, radius R, in F; and if FOG be a diameter, and AD perpendicular to BC; then

(i.) $FC=FQ=FB=2R\sin\frac{A}{2}$.

(ii.) $\angle FAD=FAO=\frac{1}{2}(B-C)$,
and $\angle CAG=\frac{1}{2}(B+C)$.



PROOF OF (i.)—
$\angle FQC = QCA+QAC.$
But $QAC = QAB = BCF$, (III. 21)
$\therefore FQC = FCQ, \quad \therefore FC = FQ.$
Similarly $FB = FQ$.
Also $\angle GCF$ is a right angle, and
$FGC = FAC = \frac{1}{2}A$, (III. 21)
$\therefore FC = 2R\sin\frac{A}{2}.$

---

936 If R, r be the radii of the circumscribed and inscribed circles of the triangle ABC (see last figure), and O, Q the centres; then $OQ^2 = R^2-2Rr.$

PROOF.—Draw QH perpendicular to AC; then $QH = r$. By the isosceles triangle AOF, $OQ^2 = R^2-AQ.QF$ (922, iii.), and $QF = FC$ (935, i.), and by similar triangles GFC, AQH, $AQ : QH :: GF : FC$,
therefore $AQ.FC = GF.QH = 2Rr.$

The problems known as the Tangencies.

937 Given in position any three of the following nine data—viz., three points, three straight lines, and three circles,—it is required to describe a circle passing through the given points and touching the given lines or circles. The following five principal cases occur.

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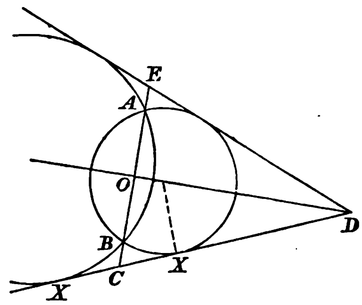
938 I. Given two points A, B, and the straight line CD.



ANALYSIS.—Let ABX be the required circle, touching CD in X. Therefore

$$CX^2 = CA \cdot CB. \qquad \text{(III. 36)}$$

Hence the point X can be found, and the centre of the circle defined by the intersection of the perpendicular to CD through X and the perpendicular bisector of AB. There are two solutions.

Otherwise, by (926), making the ratio one of equality, and DO the given line.

COR.—The point X thus determined is the point in CD at which the distance AB subtends the greatest angle. In the solution of (941) Q is a similar point in the circumference CD. (III. 21, & I. 16.)

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939 II. Given one point A and two straight lines DC, DE.

In the last figure draw AOC perpendicular to DO, the bisector of the angle D, and make $OB = OA$, and this case is solved by Case I.

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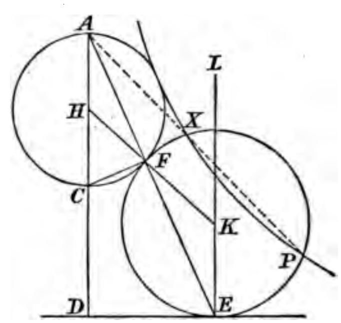
940 III. Given the point P, the straight line DE, and the circle ACF.



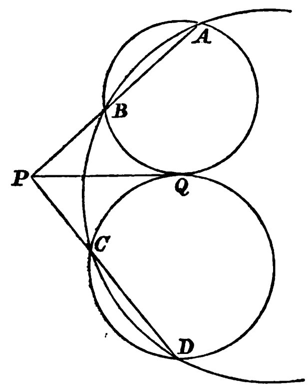
ANALYSIS.—Let PEF be the required circle touching the given line in E and the circle in F.

Through H, the centre of the given circle, draw $AHCD$ perpendicular to DE. Let K be the centre of the other circle. Join HK, passing through F, the point of contact. Join AF, EF, and AP, cutting the required circle in X. Then

$$\angle DHF = LKF, \qquad \text{(I. 27)}$$

therefore $HFA = KFE$ (the halves of equal angles), therefore AF, FE are in the same straight line. Then, because $AX \cdot AP = AF \cdot AE$, (III. 36) and $AF \cdot AE = AC \cdot AD$ by similar triangles, therefore AX can be found. A circle must then be described through P and X to touch the given line, by Case I. There are two solutions with exterior contact, as appears from Case I. These are indicated in the diagram. There are two more in which the circle AC lies within the described circle. The construction is quite analogous, C taking the place of A.

941 IV. Given two points A, B and the circle CD.

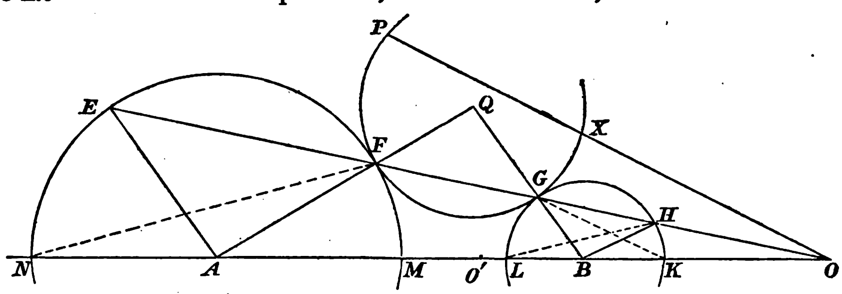


Draw any circle through A, B, cutting the ~~required~~ given circle in C, D. Draw AB and DC, and let them meet in P. Draw PQ to touch the given circle. Then, because

$$PC.PD = PA.PB = PQ^2, \quad \text{(III. 36)}$$

and the required circle is to pass through A, B; therefore a circle drawn through A, B, Q must touch PQ, and therefore the circle CD, in Q (III. 37), and it can be described by Case I. There are two solutions corresponding to the two tangents from P to the circle CD.

942 V. Given one point P, and two circles, centres A and B.



Analysis.—Let PFG be the required circle touching the given ones in F and G. Join the centres QA, QB. Join FG, and produce it to cut the circles in E and H, and the line of centres in O. Then, by the isosceles triangles, the four angles at E, F, G, H are all equal; therefore AE, BG are parallel, and so are AF, BH; therefore $AO : BO :: AF : BH$, and O is a centre of similitude for the two circles. Again, $\angle HBK = 2HLK$, and $FAM = 2FNM$ (III. 20); therefore $FNM = HLK = HGK$ (III. 21); therefore the triangles OFN, OKG are similar; therefore $OF.OG = OK.ON$; therefore, if OP cut the required circle in X, $OX.OP = OK.ON$. Thus the point X can be found, and the problem is reduced to Case IV.

Two circles can be drawn through P and X to touch the given circles. One is the circle PFX. The centre of the other is at the point where EA and HB meet if produced, and this circle touches the given ones in E and H.

943 An analogous construction, employing the internal centre of similitude O', determines the circle which passes through P, and touches one given circle externally and the other internally. See (1047–9).

The centres of similitude are the two points which divide the distance between the centres in the ratio of the radii. See (1037).

944 Cor.—The tangents from O to all circles which touch the given circles, either both externally or both internally, are equal.

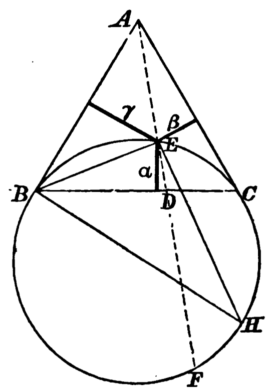
For the square of the tangent is always equal to $OK.ON$ or $OL.OM$.

945 The solutions for the cases of three given straight lines or three given points are to be found in Euc. IV., Props. 4, 5.

946 In the remaining cases of the tangencies, straight lines and circles alone are given. By drawing a circle concentric with the required one through the centre of the least given circle, the problem can always be made to depend upon one of the preceding cases; the centre of the least circle becoming one of the *given* points.

947 Definition.—*A centre of similarity of two plane curves is a point such that, any straight line being drawn through it to cut the curves, the segments of the line intercepted between the point and the curves are in a constant ratio.*

948 If AB, AC touch a circle at B and C, then any straight line $AEDF$, cutting the circle, is divided harmonically by the circumference and the chord of contact BC.



Proof from $AE.AF = AB^2$. (III. 36)

$AB^2 = BD.DC + AD^2$, (923)

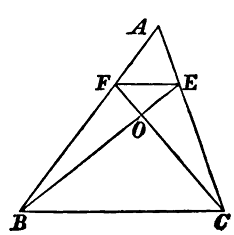
and $BD.DC = ED.DF$. (III. 35)

949 If α, β, γ, in the same figure, be the perpendiculars to the sides of ABC from any point E on the circumference of the circle, then $\beta\gamma = \alpha^2$.

Proof.—Draw the diameter $BH = d$; then $EB^2 = \beta d$, because BEH is a right angle. Similarly $EC^2 = \gamma d$. But $EB.EC = \alpha d$ (VI. D.), therefore &c.

950 If FE be drawn parallel to the base BC of a triangle, and if EB, FC intersect in O, then

$$AE : AC :: EO : OB :: FO : OC.$$



By VI. 2. Since each ratio $= FE : BC$.

Cor.—If $AC = n.AE$, then

$$BE = (n+1)\,OE.$$

951 The three lines drawn from the angles of a triangle to the middle points of the opposite sides, intersect in the same point, and divide each other in the ratio of two to one.

For, by the last theorem, any one of these lines is divided by each of the others in the ratio of two to one, measuring from the same extremity, and must therefore be intersected by them in the same point.

This point will be referred to as the *centroid* of the triangle.

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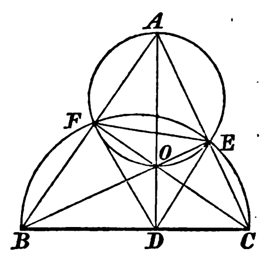
952 The perpendiculars from the angles upon the opposite sides of a triangle intersect in the same point.



Draw BE, CF perpendicular to the sides, and let them intersect in O. Let AO meet BC in D. Circles will circumscribe $AEOF$ and $BFEC$, by (III. 31);

therefore $\angle FAO = FEO = FCB$; (III. 21)

therefore $\angle BDA = BFC =$ a right angle;

i.e., AO is perpendicular to BC, and therefore the perpendicular from A on BC passes through O.

O is called the *orthocentre* of the triangle ABC.

Cor.—The perpendiculars on the sides bisect the angles of the triangle DEF, and the point O is therefore the centre of the inscribed circle of that triangle.

Proof.—From (III. 21), and the circles circumscribing $OEAF$ and $OECD$.

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953 If the inscribed circle of a triangle ABC touches the sides a, b, c in the points D, E, F; and if the escribed circle to the side a touches a and b, c produced in D', E', F'; and if

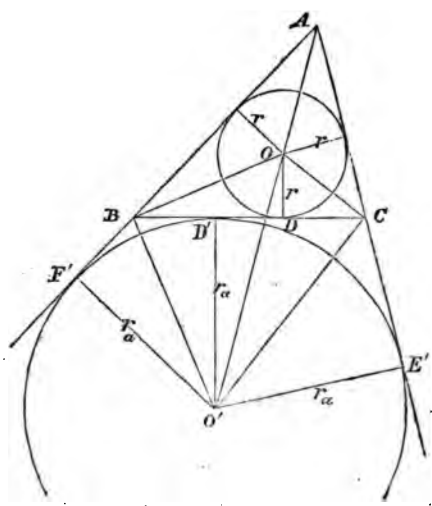
$$s = \tfrac{1}{2}(a+b+c);$$

then

$$BF' = BD' = CD = s-c,$$

and $AE' = AF' = s$;

and similarly with respect to the other segments.



Proof.—The two tangents from any vertex to either circle being equal, it follows that $CD+c =$ half the perimeter of ABC, which is made up of three pairs of equal segments;

therefore $CD = s-c$.

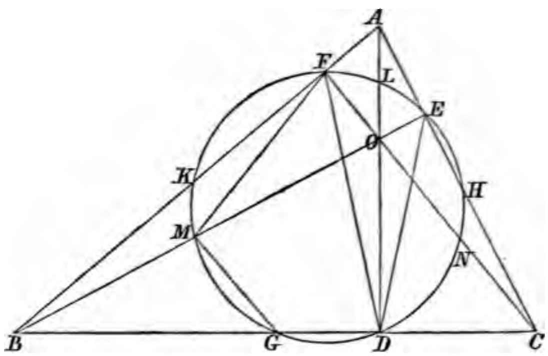
Also

$$AE'+AF' = AC+CD'+AB+BD' = 2s;$$

therefore $AE' = AF' = s$.

The Nine-Point Circle.

954 The *Nine-point* circle is the circle described through D, E, F, the feet of the perpendiculars on the sides of the triangle ABC. It also passes through the middle points of the sides of ABC and the middle points of OA, OB, OC; in all, through nine points.

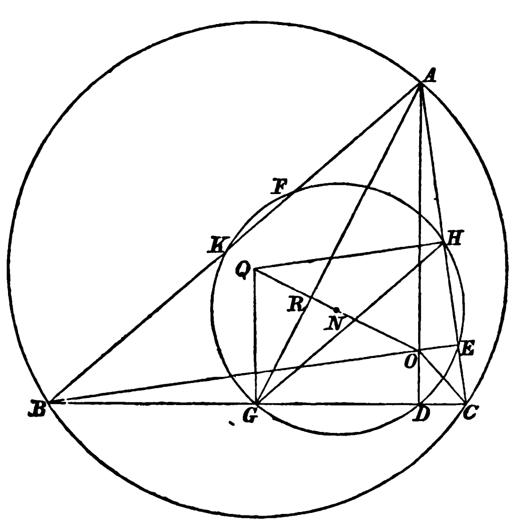


PROOF.—Let the circle cut the sides of ABC again in G, H, K; and OA, OB, OC in L, M, N. $\angle EMF = EDF$ (III. 21) $= 2ODF$ (952, Cor.); therefore, since OB is the diameter of the circle circumscribing $OFBD$ (III. 31), M is the centre of that circle (III. 20), and therefore bisects OB.

Similarly OC and OA are bisected at N and L.

Again, $\angle MGB = MED$ (III. 22) $= OCD$, (III. 21), by the circle circumscribing $OECD$. Therefore MG is parallel to OC, and therefore bisects BC. Similarly H and K bisect CA and AB.

955 The centre of the nine-point circle is the middle point of OQ, the line joining the orthocentre and the centre of the circumscribing circle of the triangle ABC.



For the centre of the N. P. circle is the intersection of the perpendicular bisectors of the chords DG, EH, FK, and these perpendiculars bisect OQ in the same point N, by (VI. 2).

956 The centroid of the triangle ABC also lies on the line OQ and divides it in R so that $OR = 2RQ$.

PROOF.—The triangles QHG, OAB are similar, and $AB = 2HG$; therefore $AO = 2GQ$; therefore $OR = 2RQ$; and $AR = 2RG$; therefore R is the centroid, and it divides OQ as stated (951).

957 Hence the line joining the centres of the circumscribed and nine-point circles is divided harmonically in the ratio of 2 : 1 by the centroid and the orthocentre of the triangle.

These two points are therefore centres of similitude of the circumscribed and nine-point circles; and any line drawn through either of the points is divided by the circumferences in the ratio of 2 : 1. See (1037.)

958 The lines DE, EF, FD intersect the sides of ABC in the radical axis of the two circles.

For, if EF meets BC in P, then by the circle circumscribing $BCEF$, $PE.PF = PC.PB$; therefore (III. 36) the tangents from P to the circles are equal (985).

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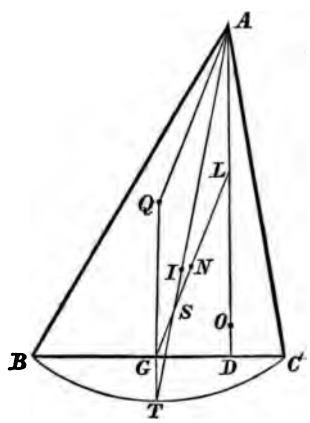
959 The nine-point circle touches the inscribed and escribed circles of the triangle.



PROOF.—Let O be the orthocentre, and I, Q the centres of the inscribed and circumscribed circles. Produce AI to bisect the arc BC in T. Bisect AO in L, and join GL, cutting AT in S.

The N. P. circle passes through G, D, and L (954), and D is a right angle. Therefore GL is a diameter, and is therefore $= R = QA$ (957). Therefore GL and QA are parallel. But $QA = QT$, therefore

$$GS = GT = CT \sin\frac{A}{2} = 2R\sin^2\frac{A}{2}. \quad (935, \text{i.})$$

Also $\qquad ST = 2GS\cos\theta$

(θ being the angle $GST = GTS$).

N being the centre of the N. P. circle, its radius $= NG = \frac{1}{2}R$; and r being the radius of the inscribed circle, it is required to shew that

$$NI = NG - r.$$

Now $\qquad NI^2 = SN^2 + SI^2 - 2SN.SI\cos\theta. \qquad (702)$

Substitute $\qquad SN = \frac{1}{2}R - GS;$

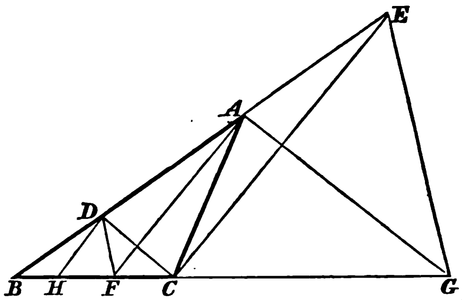
$$SI = TI - ST = 2R\sin\frac{A}{2} - 2GS\cos\theta;$$

and $GS = 2R\sin^2\frac{1}{2}A$, to prove the proposition.

If J be the centre of the escribed circle touching BC, and r_a its radius, it is shewn in a similar way that $NJ = NG + r_a$.

To construct a triangle from certain data.

960 When amongst the data we have the sum or difference of the two sides AB, AC; or the sum of the segments of the base made by AG, the bisector of the exterior vertical angle; or the difference of the segments made by AF', the bisector of the interior vertical angle; the following construction will lead to the solution.



Make $AE = AD = AC$. Draw DH parallel to AF, and suppose EK drawn parallel to AG to meet the base produced in K; and complete the figure. Then BE is the sum, and BD is the difference of the sides.

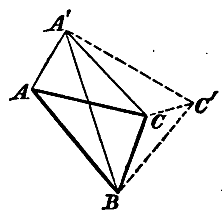
BK is the sum of the exterior segments of the base, and BH is the difference of the interior segments. $\angle BDH = BEC = \frac{1}{2}A$,

$\angle ADC = EAG = \frac{1}{2}(B+C)$,

$\angle DCB = \frac{1}{2}DFB = \frac{1}{2}(C-B)$.

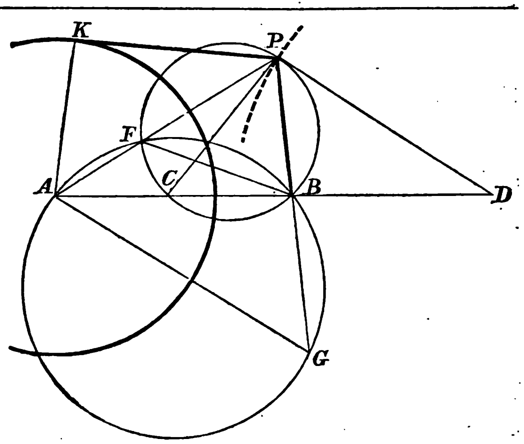
961 When the base and the vertical angle are given; the locus of the vertex is the circle ABC in figure (935); and the locus of the centre of the inscribed circle is the circle, centre F and radius FB. When the ratio of the sides is given, see (932).

962 To construct a triangle when its form and the distances of its vertices from a point A' are given.



ANALYSIS.—Let ABC be the required triangle. On $A'B$ make the triangle $A'BC'$ similar to ABC, so that $AB : A'B :: CB : C'B$. The angles ABA', CBC' will also be equal; therefore $AB : BC :: AA' : CC'$, which gives CC', since the ratio $AB : BC$ is known. Hence the point C is found by constructing the triangle $A'CC'$. Thus BC is determined, and thence the triangle ABC from the known angles.

963 To find the locus of a point P, the tangent from which to a given circle, centre A, has a constant ratio to its distance from a given point B.



Let AK be the radius of the circle, and $p : q$ the given ratio. On AB take AC, a third proportional to AB and AK, and make

$$AD : DB = p^2 : q^2.$$

With centre D, and a radius

equal to a mean proportional between DB and DC, describe a circle. It will be the required locus.

PROOF.—Suppose P to be a point on the required locus. Join P with A, B, C, and D.

Describe a circle about PBC cutting AP in F, and another about ABF cutting PB in G, and join AG and BF. Then

$$PK^2 = AP^2 - AK^2 = AP^2 - BA.AC \text{ (by constr.)} = AP^2 - PA.AF \text{ (III. 36)}$$
$$= AP.PF \text{ (II. 2)} = GP.PB \text{ (III. 36)}.$$

Therefore, by hypothesis,

$$p^2 : q^2 = GP.PB : PB^2 = GP : PB = AD : DB \text{ (by constr.)};$$

therefore $\angle DPG = PGA$ (VI. 2) $= PFB$ (III. 22) $= PCB$ (III. 21).

Therefore the triangles DPB, DCP are similar; therefore DP is a mean proportional to DB and DC. Hence the construction.

964 COR.—If $p = q$ the locus becomes the perpendicular bisector of BC, as is otherwise shown in (1003).

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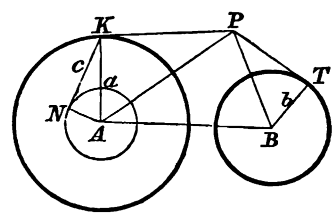
965 To find the locus of a point P, the tangents from which to two given circles shall have a given ratio. (See also 1036.)



Let A, B be the centres, a, b the radii ($a > b$), and $p : q$ the given ratio. Take c, so that $c : b = p : q$, and describe a circle with centre A and radius $AN = \sqrt{a^2 - c^2}$. Find the locus of P by the last proposition, so that the tangent from P to this circle may have the given ratio to PB. It will be the required locus.

PROOF.—By hypothesis and construction,

$$\frac{p^2}{q^2} = \frac{PK^2}{PT^2} = \frac{c^2}{b^2} = \frac{PK^2 + c^2}{PT^2 + b^2} = \frac{AP^2 - a^2 + c^2}{BP^2} = \frac{AP^2 - AN^2}{BP^2}.$$

COR.—Hence the point can be found on any curve from which the tangents to two circles shall have a given ratio.

---

966 To find the locus of the point from which the tangents to two given circles are equal.

Since, in (965), we have now $p = q$, and therefore $c = b$, the construction simplifies to the following:

Take $AN = \sqrt{(a^2 - b^2)}$, and in AB take $AB : AN : AC$. The perpendicular bisector of BC is the required locus. But, if the circles intersect, then their common chord is at once the line required. See Radical Axis (985).

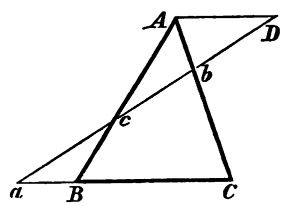
Collinear and Concurrent systems of points and lines.

967 Definitions.—Points lying in the same straight line are *collinear*. Straight lines passing through the same point are *concurrent*, and the point is called the *focus* of the pencil of lines.

Theorem.—If the sides of the triangle ABC, or the sides produced, be cut by any straight line in the points a, b, c respectively, the line is called a *transversal*, and the segments of the sides are connected by the equation

968 $$(Ab : bC)\,(Ca : aB)\,(Bc : cA) = 1.$$

Conversely, if this relation holds, the points a, b, c will be collinear.



Proof.—Through any vertex A draw AD parallel to the opposite side BC, to meet the transversal in D, then

$Ab : bC = AD : Ca$ and $Bc : cA = aB : AD$

(VI. 4), which proves the theorem.

Note.—In the formula the segments of the sides are estimated positive, independently of direction, the sequence of the letters being preserved the better to assist the memory. A point may be supposed to travel from A over the segments Ab, bC, &c. *continuously*, until it reaches A again.

969 By the aid of (701) the above relation may be put in the form

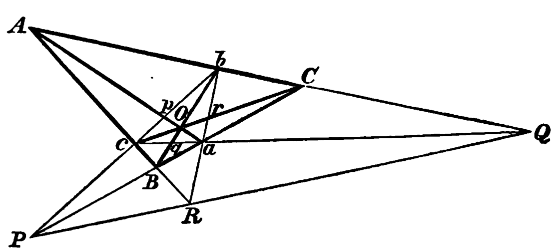
$$(\sin ABb : \sin bBC)(\sin CAa : \sin aAB)(\sin BCc : \sin cCA) = 1$$

---

970 If O be any focus in the plane of the triangle ABC, and if AO, BO, CO meet the sides in a, b, c; then, as before,

$$(Ab : bC)\,(Ca : aB)\,(Bc : cA) = 1.$$

Conversely, if this relation holds, the lines Aa, Bb, Cc will be concurrent.



Proof.—By the transversal Bb to the triangle AaC, we have (968)

$(Ab : bC)\,(CB : Ba)$
$\times (aO : OA) = 1.$

And, by the transversal Cc to the triangle AaB,

$(Bc : cA)\,(AO : Oa)$
$\times (aC : CB) = 1.$

Multiply these equations together.

971 If bc, ca, ab, in the last figure, be produced to meet the sides of ABC in P, Q, R, then each of the nine lines in the figure will be divided harmonically, and the points P, Q, R will be collinear.

PROOF.—(i.) Take bP a transversal to ABC; therefore, by (968),

$$(CP : PB)\,(Bc : cA)\,(Ab : bC) = 1;$$

therefore, by (970), $\quad CP : PB = Ca : aB.$

(ii.) Take CP a transversal to Abc, therefore

$$(AB : Bc)\,(cP : Pb)\,(bC : CA) = 1.$$

But, by (970), taking O for focus to Abc,

$$(AB : Bc)\,(cp : pb)\,(bC : CA) = 1;$$

therefore $\quad cP : Pb = cp : pb.$

(iii.) Take PC a transversal to AOc, and b a focus to AOc; therefore, by (968 & 970), $\quad (Aa : aO)\,(OC : Cc)\,(cB : BA) = 1,$

and $\quad (Ap : pO)\,(OC : Cc)\,(cB : BA) = 1;$

therefore $\quad Aa : aO = Ap : pO.$

Thus all the lines are divided harmonically.

(iv.) In the equation of (970) put $Ab : bC = AQ : QC$ the harmonic ratio, and similarly for each ratio, and the result proves that P, Q, R are collinear, by (968).

COR.—If in the same figure qr, rp, pq be joined, the three lines will pass through P, Q, R respectively.

PROOF.—Take O as a focus to the triangle abc, and employ (970) and the harmonic division of bc to show that the transversal rq cuts bc in P.

---

972 If a transversal intersects the sides AB, BC, CD, &c. of any polygon in the points a, b, c, &c. in order, then

$$(Aa : aB)\,(Bb : bC)\,(Cc : cD)\,(Dd : dE)\ \dots\ \&c. = 1.$$

PROOF.—Divide the polygon into triangles by lines drawn from one of the angles, and, applying (968) to each triangle, combine the results.

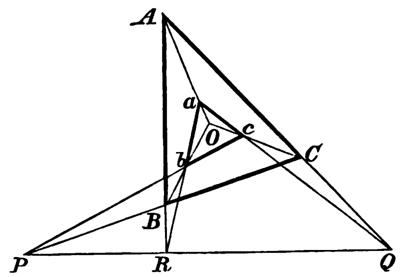
---

973 Let any transversal cut the sides of a triangle and their three intersectors AO, BO, CO (see figure of 970) in the points A', B', C', a', b', c', respectively; then, as before,

$$(A'b' : b'C')\,(C'a' : a'B')\,(B'c' : c'A') = 1.$$

PROOF.—Each side forms a triangle with its intersector and the transversal. Take the four remaining lines in succession for transversals to each triangle, applying (968) symmetrically, and combine the twelve equations.

974 If the lines joining corresponding vertices of two triangles ABC, abc are concurrent, the points of intersection of the pairs of corresponding sides are collinear, and conversely.



PROOF.—Let the concurrent lines Aa, Bb, Cc meet in O. Take bc, ca, ab transversals respectively to the triangles OBC, OCA, OAB, applying (968), and the product of the three equations shows that P, R, Q lie on a transversal to ABC.

975 Hence it follows that, if the lines joining each pair of corresponding vertices of any two rectilineal figures are concurrent, the pairs of corresponding sides intersect in points which are collinear.

The figures in this case are said to be *in perspective*, or *in homology*, with each other. The point of concurrence and the line of collinearity are called respectively the *centre* and *axis* of perspective or homology. See (1083).

976 *Theorem.*—When three perpendiculars to the sides of a triangle ABC, intersecting them in the points a, b, c respectively, are *concurrent*, the following relation is satisfied; and conversely, if the relation be satisfied, the perpendiculars are concurrent.

$$Ab^2 - bC^2 + Ca^2 - aB^2 + Bc^2 - cA^2 = 0.$$

PROOF.—If the perpendiculars meet in O, then $Ab^2 - bC^2 = AO^2 - OC^2$, &c. (I. 47).

EXAMPLES.—By the application of this theorem, the concurrence of the three perpendiculars is readily established in the following cases:—

(1) When the perpendiculars bisect the sides of the triangle

(2) When they pass through the vertices. (By employing I. 47.)

(3) The three radii of the escribed circles of a triangle at the points of contact between the vertices are concurrent. So also are the radius of the inscribed circle at the point of contact with one side, and the radii of the two escribed circles of the remaining sides at the points of contact beyond the included angle.

In these cases employ the values of the segments given in (953).

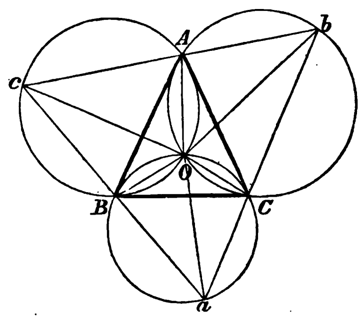
(4) The perpendiculars equidistant from the vertices with three concurrent perpendiculars are also concurrent.

(5) When the three perpendiculars from the vertices of one triangle upon the sides of the other are concurrent, then the perpendiculars from the vertices of the second triangle upon the sides of the first are also concurrent.

PROOF.—If A, B, C and A', B', C' are corresponding vertices of the triangles, join AB', AC', BC', BA', CA', CB', and apply the theorem in conjunction with (I. 47).

Triangles of constant species circumscribed to a triangle.

977 Let ABC be any triangle, and O any point; and let circles circumscribe AOB, BOC, COA. The circumferences will be the loci of the vertices of a triangle of constant form whose sides pass through the points A, B, C.



PROOF.—Draw any line bAc from circle to circle, and produce bC, cB to meet in a. The angles AOB, COA are supplements of the angles c and b (III. 22); therefore BOC is the supplement of a (I. 32); therefore a lies on the circle OBC. Also, the angles at O being constant, the angles a, b, c are constant.

978 The triangle abc is a maximum when its sides are perpendicular to OA, OB, OC.

PROOF.—The triangle is greatest when its sides are greatest. But the sides vary as Oa, Ob, Oc, which are greatest when they are diameters of the circles; therefore &c., by (III. 31).

979 To construct a triangle of given species and of given limited magnitude which shall have its sides passing through three given points A, B, C.

Determine O by describing circles on the sides of ABC to contain angles equal to the supplements of the angles of the specified triangle. Construct the figure $abcO$ independently from the known sides of abc, and the now known angles $ObC = OAC$, $OaC = OBC$, &c. Thus the lengths Oa, Ob, Oc are found, and therefore the points a, b, c, on the circles, can be determined.

The demonstrations of the following propositions will now be obvious.

Triangles of constant species inscribed to a triangle.

980 Let abc, in the last figure, be a fixed triangle, and O any point. Take any point A on bc, and let the circles circumscribing OAc, OAb cut the other sides in B, C. Then ABC will be a triangle of constant form, and its angles will have the values $A = Oba + Oca$, &c. (III. 21.)

981 The triangle ABC will evidently be a minimum when OA, OB, OC are drawn perpendicular to the sides of abc.

982 To construct a triangle of given form and of given limited magnitude having its vertices upon three fixed lines bc, ca, ab.

Construct the figure $ABCO$ independently from the known sides of ABC and the angles at O, which are equal to the supplements of the given angles a, b, c. Thus the angles OAC, &c. are found, and therefore the angles ObC, &c., equal to them (III. 21), are known. From these last angles the point O can be determined, and the lengths OA, OB, OC being known from the independent figure, the points A, B, C can be found.

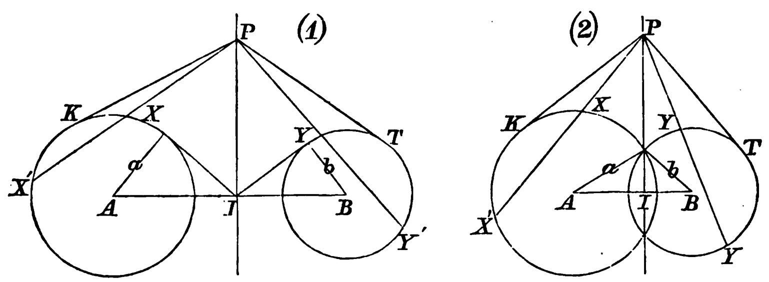
Observe that, wherever the point O may be taken, the angles AOB, BOC COA are in all cases either the supplements of, or equal to, the angles c, a, b respectively; while the angles aOb, bOc, cOa are in all cases equal to $C \pm c$, $A \pm a$, $B \pm b$.

983 NOTE.—In general problems, like the foregoing, which admit of different cases, it is advisable to choose for reference a standard figure which has all its elements of the same affection or sign. In adapting the figure to other cases, all that is necessary is to follow the same construction, letter for letter, observing the convention respecting positive and negative, which applies both to the lengths of lines and to the magnitudes of angles, as explained in (607—609).

Radical Axis.

984 DEFINITION.—The *radical axis* of two circles is that perpendicular to the line of centres which divides the distance between the centres into segments, the difference of whose squares is equal to the difference of the squares of the radii.

Thus, A, B being the centres, a, b the radii, and IP the the radical axis, $AI^2 - BI^2 = a^2 - b^2$.



985 It follows that, if the circles intersect, the radical axis is their common chord; and that, if they do not intersect, the radical axis cuts the line of centres in a point the tangents from which to the circles are equal (I. 47).

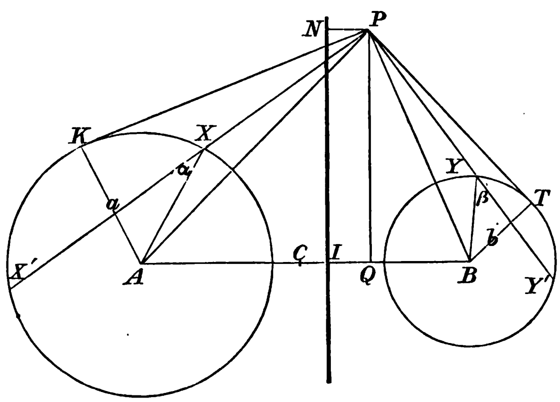
To draw the axis in this case, see (966).

Otherwise: let the two circles cut the line of centres in C, D and C', D' respectively. Describe any circle through C and D, and another through C' and D', intersecting the former in E and F. Their common chord EF will cut the central axis in the required point I.

Proof.—$IC.ID = IE.IF = IC'.ID'$ (III. 36); therefore the tangents from I to the circles are equal.

986 *Theorem.*—The difference of the squares of tangents from any point P to two circles is equal to twice the rectangle under the distance between their centres and the distance of the point from their radical axis, or

$$PK^2 - PT^2 = 2AB \cdot PN.$$



Proof.

$$PK^2 - PT^2 = (AP^2 - BP^2) - (a^2 - b^2) = (AQ^2 - BQ^2) - (AI^2 - BI^2),$$

by (I. 47) & (984). Bisect AB in C, and substitute for each difference of squares, by (II. 12).

987 Cor. 1.—If P be on the circle whose centre is B, then

$$PK^2 = 2AB \cdot PN.$$

988 Cor. 2.—If two chords be drawn through P to cut the circles in X, X', Y, Y'; then, by (III. 36),

$$PX \cdot PX' - PY \cdot PY' = 2AB \cdot PN.$$

989 If a variable circle intersect two given circles at constant angles α and β, it will intersect their radical axis at a constant angle; and its radius will bear a constant ratio to the distance of its centre from the radical axis. Or

$$PN : PX = a \cos\alpha - b \cos\beta : AB.$$

PROOF.—In the same figure, if P be the centre of the variable circle, and if $PX = PY$ be its radius; then, by (988),

$$PX\,(XX' - YY') = 2AB\,.\,PN.$$

But $\qquad XX' = 2a\cos\alpha$ and $YY' = 2b\cos\beta$;

therefore $\qquad PN : PX = a\cos\alpha - b\cos\beta : AB,$

which is a constant ratio if the angles α, β are constant.

990 Also $PX : PN =$ the cosine of the angle at which the circle of radius PX cuts the radical axis. This angle is therefore constant.

991 COR.—A circle which touches two fixed circles has its radius in a constant ratio to the distance of its centre from their radical axis.

This follows from the proposition by making $\alpha = \beta = 0$ or 2π.

If P be on the radical axis; then (see Figs. 1 and 2 of 984)

992 (i.) The tangents from P to the two circles are equal, or $\qquad PK = PT. \qquad$ (986)

993 (ii.) The rectangles under the segments of chords through P are equal, or $PX\,.\,PX' = PY\,.\,PY'$. $\qquad$ (988)

994 (iii.) Therefore the four points X, X', Y, Y' are concyclic (III. 36); and, conversely, if they are concyclic, the chords XX', YY' intersect in the radical axis.

995 DEFINITION.—Points which lie on the circumference of a circle are termed *concyclic*.

996 (iv.) If P be the centre, and if $PX = PY$ be the radius of a circle intersecting the two circles in the figure at angles α and β; then, by (993), $XX' = YY'$, or $a\cos\alpha = b\cos\beta$; that is, *The cosines of the angles of intersection are inversely as the radii of the fixed circles.*

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997 The radical axes of three circles (Fig. 1046), taken two and two together, intersect at a point called their *radical centre.*

PROOF.—Let A, B, C be the centres, a, b, c the radii, and X, Y, Z the points in which the radical axes cut BC, CA, AB. Write the equation of the definition (984) for each pair of circles. Add the results, and apply (976).

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998 A circle whose centre is the radical centre of three other circles intersects them in angles whose cosines are inversely as their radii (996).

Hence, if this fourth circle cuts one of the others orthogonally, it cuts them all orthogonally.

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999 The circle which intersects at angles α, β, γ three fixed circles, whose centres are A, B, C and radii a, b, c, has its centre at distances from the radical axes of the fixed circles proportional to

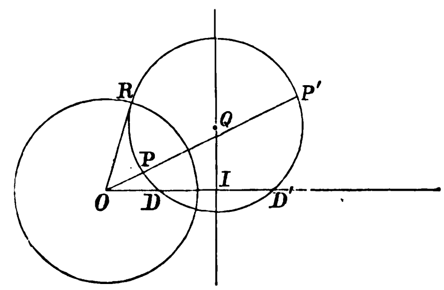
$$\frac{b\cos\beta - c\cos\gamma}{BC}, \quad \frac{c\cos\gamma - a\cos\alpha}{CA}, \quad \frac{a\cos\alpha - b\cos\beta}{AB}.$$

And therefore the locus of its centre will be a straight line passing through the radical centre and inclined to the three radical axes at angles whose sines are proportional to these fractions.

PROOF.—The result is obtained immediately by writing out equation (989) for each pair of fixed circles.

The Method of Inversion.

1000 DEFINITIONS.—Any two points P, P', situated on a diameter of a fixed circle whose centre is O and radius k, so that $OP . OP' = k^2$, are called *inverse points* with respect to the circle, and either point is said to be the *inverse* of the other. The circle and its centre are called the *circle and centre of inversion*, and k the *constant of inversion.*



1001 If every point of a plane figure be inverted with respect to a circle, or every point of a figure in space with respect to a sphere, the resulting figure is called the *inverse* or *image* of the original one.

Since $OP : k : OP'$, therefore

1002 $$OP : OP' = OP^2 : k^2 = k^2 : OP'^2.$$

1003 Let D, D', in the same figure, be a pair of inverse points on the diameter OD'. In the perpendicular bisector of DD', take any point Q as the centre of a circle passing through D, D', cutting the circle of inversion in R, and any straight line through O in the points P, P'. Then, by (III. 36), $OP . OP' = OD . OD' = OR^2$ (1000). Hence

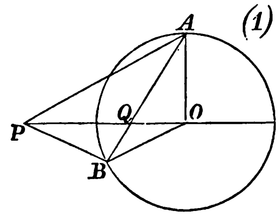
1004 (i.) P, P' are inverse points; and, conversely, any two pairs of inverse points lie on a circle.

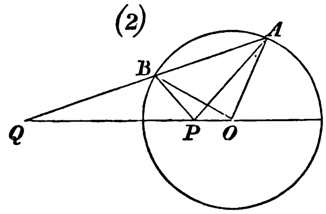
1005 (ii.) The circle cuts orthogonally the circle of inversion (III. 37); and, conversely, every circle cutting another orthogonally intersects each of its diameters in a pair of inverse points.

1006 (iii.) The line IQ is the locus of a point the tangent from which to a given circle is equal to its distance from a given point D.

1007 Def.—The line IQ is called the *axis of reflexion* for the two inverse points D, D', because there is another circle of inversion, the reflexion of the former, to the right of IQ, having also D, D' for inverse points.

1008 The straight lines drawn from any point P, within or without a circle (Figs. 1 and 2), to the extremities of any chord AB passing through the inverse point Q, make equal angles with the diameter through PQ. Also, the four points O, A, B, P are concyclic, and $QA \cdot QB = QO \cdot QP$.





Proof.—In either figure $OP : OA : OQ$ and $OP : OB : OQ$ (1000), therefore, by similar triangles, $\angle OPA = OAB$ and $OPB = OBA$ in figure (1) and the supplement of it in figure (2). But $OAB = OBA$ (I. 5), therefore, &c.

Also, because $\angle OPA = OBA$, the four points O, A, B, P lie on a circle in each case (III. 21), and therefore $QA \cdot QB = QO \cdot QP$ (III. 35, 36).

1009 The inverse of a circle is a circle, and the centre of inversion is the centre of similitude of the two figures. See also (1037).

Proof.—In the figure of (1043), let O be the point where the common tangent RT of the two circles, centres A and B, cuts the central axis, and let any other line through O cut the circles in P, Q P', Q'. Then, in the demonstration of (942), it is shown that $OP \cdot OQ' = OQ \cdot OP' = k^2$, a constant quantity. Therefore either circle is the inverse of the other, k being the radius of the circle of inversion.

1010 To make the inversions of two given circles equal circles.

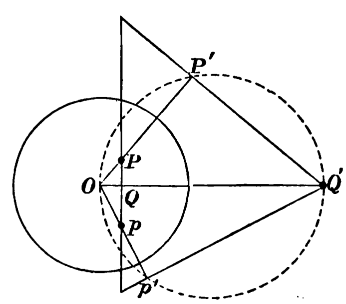
RULE.—*Take the centre of inversion so that the squares of the tangents from it to the given circles may be proportional to their radii* (965).

PROOF.—(Fig. 1043) $AT : BR = OT : OR = OT^2 : k^2$, since $OT : k : OR$. Therefore $OT^2 : AT = k^2 : BR$, therefore BR remains constant if $OT^2 \propto AT$.

1011 Hence three circles may be inverted into equal circles, for the required centre of inversion is the intersection of two circles that can be drawn by (965).

1012 The inverse of a straight line is a circle passing through the centre of inversion.

PROOF.—Draw OQ perpendicular to the line, and take P any other point on it. Let Q', P' be the inverse points. Then $OP \cdot OP' = OQ \cdot OQ'$; therefore, by similar triangles, $\angle OP'Q' = OQP$, a right angle; and OQ' is constant, therefore the locus of P' is the circle whose diameter is OQ'.



1013 EXAMPLE.—The inversion of a polygon produces a figure bounded by circular arcs which intersect in angles equal to the corresponding angles of the polygon, the complete circles intersecting in the centre of inversion.

1014 If the extremities of a straight line $P'Q'$ in the last figure are the inversions of the extremities of PQ, then

$$PQ : P'Q' = \sqrt{(OP \cdot OQ)} : \sqrt{(OP' \cdot OQ')}.$$

PROOF.—By similar triangles, $PQ : P'Q' = OP : OQ'$ and $PQ : P'Q' = OQ : OP'$. Compound these ratios.

1015 From the above it follows that any homogeneous equation between the lengths of lines joining pairs of points in space, such as $PQ \cdot RS \cdot TU = PR \cdot QT \cdot SU$, the same points appearing on both sides of the equation, will be true for the figure obtained by joining the corresponding pairs of inverse points.

For the ratio of each side of the equation to the corresponding side of the equation for the inverted points will be the same, namely,

$$\sqrt{(OP \cdot OQ \cdot OR \, ...)} : \sqrt{(OP' \cdot OQ' \cdot OR' \, ...)}.$$

Pole and Polar.

1016 Definition.—The *polar* of any point P with respect to a circle is the perpendicular to the diameter OP (Fig. 1012) drawn through the inverse point P'.

1017 It follows that the polar of a point exterior to the circle is the *chord of contact* of the tangents fron the point; that is, the line joining their points of contact.

1018 Also, $P'Q'$ is the polar of P with respect to the circle, centre O, and PQ is the polar of Q'. In other words, *any point* P *lying on the polar of a point* Q′, *has its own polar always passing through* Q′.

1019 The line joining any two points P, p is the polar of Q', the point of intersection of their polars.

Proof.—The point Q' lies on both the lines $P'Q'$, $p'Q'$, and therefore has its polar passing through the pole of each line, by the last theorem.

1020 The polars of any two points P, p, and the line joining the points form a *self-reciprocal* triangle with respect to the circle, the three vertices being the poles of the opposite sides. The centre O of the circle is evidently the orthocentre of the triangle (952). The circle and its centre are called the *polar circle* and *polar centre* of the triangle.

If the radii of the polar and circumscribed circles of a triangle ABC be r and R, then

$$r^2 = 4R^2 \cos A \cos B \cos C.$$

Proof.—In Fig. (952), O is the centre of the polar circle, and the circles described round ABC, BOC, COA, AOB are all equal; because the angle BOC is the supplement of A; &c. Therefore $2R \,.\, OD = OB \,.\, OC$ (VI. C) and $r^2 = OA \,.\, OD = OA \,.\, OB \,.\, OC \div 2R$. Also, $OA = 2R \cos A$ by a diameter through B, and (III. 21).

Coaxal Circles.

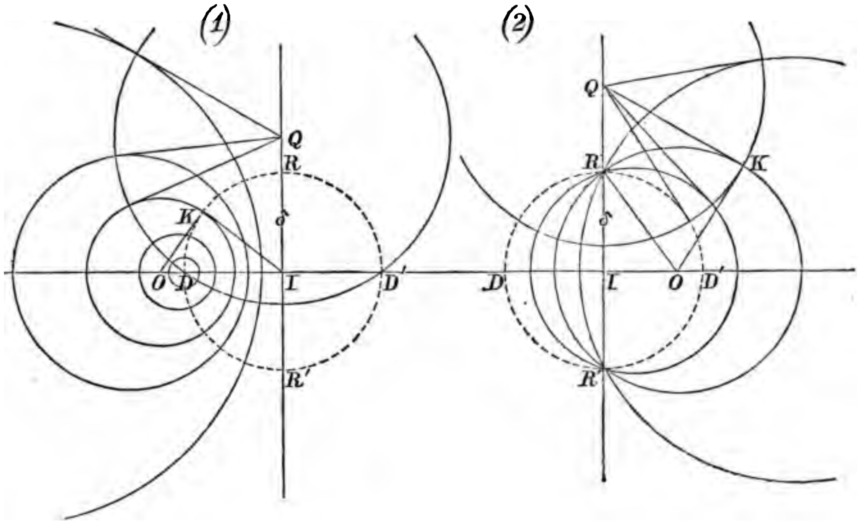
1021 Definition.—A system of circles having a common line of centres called the *central axis*, and a common radical axis, is termed a *coaxal* system.

1022 If O be the variable centre of one of the circles, and

OK its radius, the whole system is included in the equation

$$OI^2 - OK^2 = \pm \delta^2,$$

where δ is a constant length.



1023 In the first species (Fig. 1),

$$OI^2 - OK^2 = \delta^2,$$

and δ is the length of the tangent from I to any circle of the system (985). Let a circle, centre I and radius δ, cut the central axis in D, D'. When O is at D or D', the circle whose radius is OK vanishes. When O is at an infinite distance, the circle developes into the radical axis itself and into a line at infinity.

The points D, D' are called the *limiting points.*

1024 In the second species (Fig. 2),

$$OR^2 - OI^2 = \delta^2,$$

and δ is half the chord RR' common to all the circles of the system. These circles vary between the circle with centre I and radius δ, and the circle with its centre at infinity as described above. The points R, R' are the *common points* of all circles of this system. The two systems are therefore distinguished as the *limiting points species* and the *common points species* of coaxal circles.

1025 There is a conjugate system of circles having R, R' for limiting points, and D, D' for common points, and the circles of one species intersect all the circles of the conjugate system of the other species orthogonally (1005).

Thus, in figures (1) and (2), Q is the centre of a circle of the opposite species intersecting the other circles orthogonally.

1026 In the first species of coaxal circles, the limiting points D, D' are *inverse* points for every circle of the system, the radical axis being the axis of reflexion for the system.

PROOF.—(Fig. 1) $OI^2 - \delta^2 = OK^2$,

therefore $OD \,.\, OD' = OK^2$, (II. 13)

therefore D, D' are inverse points (1000).

1027 Also, the points in which any circle of the system cuts the central axis are inverse points for the circle whose centre is I and radius δ. [PROOF.—Similar to the last.

1028 PROBLEM.—Given two circles of a coaxal system, to describe a circle of the same system—(i.) to pass through a given point; or (ii.) to touch a given circle; or (iii.) to cut a given circle orthogonally.

1029 I. If the system be of the common points species, then, since the required circle always passes through two known points, the first and second cases fall under the Tangencies. See (941).

1030 To solve the third case, describe a circle through the given common points, and through the inverse of either of them with respect to the given circle, which will then be cut orthogonally, by (1005).

1031 II. If the system be of the limiting points species, the problem is solved in each case by the aid of a circle of the conjugate system. Such a circle always passes through the known limiting points, and may be called a conjugate circle of the limiting points system. Thus,

1032 To solve case (i.)—Draw a conjugate circle through the given point, and the tangent to it at that point will be the radius of the required circle.

1033 To solve case (ii.)—Draw a conjugate circle through the inverse of either limiting point with respect to the given circle, which will thus be cut orthogonally, and the tangent to the cutting circle at either point of intersection will be the radius of the required circle.

1034 To solve case (iii.)—Draw a conjugate circle to touch the given one, and the common tangent of the two will be the radius of the required circle.

1035 Thus, according as we wish to make a circle of the system *touch, or cut orthogonally*, the given circle, we must draw a conjugate circle to *cut orthogonally, or touch it.*

1036 If three circles be coaxal, the squares of the tangents drawn to any two of them from a point on the third are in the ratio of the distances of the centre of the third circle from the centres of the other two.

PROOF.—Let A, B, C be the centres of the circles; PK, PT the tangents from a point P on the circle, centre C, to the other two; PN the perpendicular on the radical axis. By (986),

$$PK^2 = 2AC.PN \quad \text{and} \quad PT^2 = 2BC.PN,$$

therefore $$PK^2 : PT^2 = AC : BC.$$

Centres and axes of similitude.

1037 DEFINITIONS.—Let OO' be the centres of similitude (Def. 947) of the two circles in the figure below, and let any line through O cut the circles in P, Q, P', Q'. Then the constant ratio $OP : OP' = OQ : OQ'$ is called the ratio of *similitude* of the two figures; and the constant product $OP.OQ' = OQ.OP'$ is called the product of *anti-similitude*. See (942), (1009), and (1043).

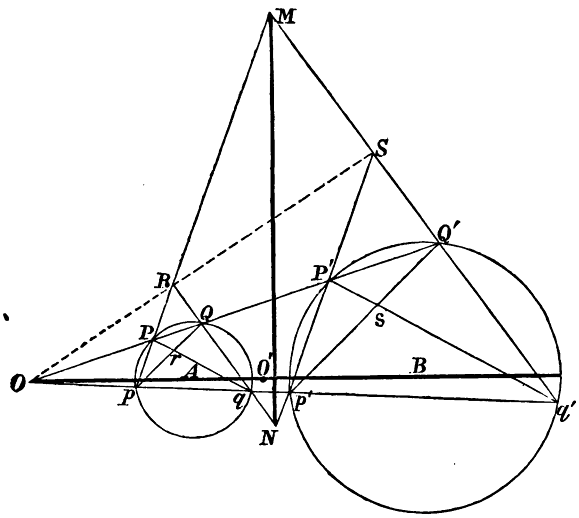
The corresponding points P, P' or Q, Q' on the same straight line through O are termed *homologous*, and P, Q' or Q, P' are termed *anti-homologous*.

1038 Let any other line $Opqp'q'$ be drawn through O. Then, if any two points P, p on the one figure be joined, and if P', p', homologous to P, p on the other figure, be also joined, the lines so formed are termed *homologous*. But if the points which are joined on the second figure are anti-homologous to those on the first, the two lines are termed *anti-homologous*. Thus, Pq, $Q'p'$ are anti-homologous lines.

1039 The circle whose centre is O, and whose radius is equal to the square root of the product of anti-similitude, is called the *circle of anti-similitude*.

1040 The four pairs of homologous chords Pp and $P'p'$, Qq and $Q'q'$, Pq and $P'q'$, Qp and $Q'p'$ of the two circles in the figure are parallel. And in all similar and similarly situated figures homologous lines are parallel.

PROOF.—By (VI. 2) and the definition (947).



1041 The four pairs of anti-homologous chords, Pp and $Q'q'$, Qq and $P'p'$, Pq and $Q'p'$, Qp and $P'q'$, of the two circles meet on their radical axis.

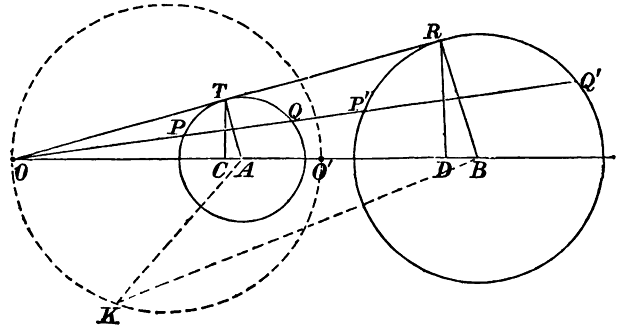
PROOF.— $OP \,.\, OQ' = Op \,.\, Oq' = k^2$,

where k is the constant of inversion; therefore P, p, Q', q' are concyclic; therefore Pp and $Q'q'$ meet on the radical axis. Similarly for any other pair of anti-homologous chords.

1042 COR.—From this and the preceding proposition it follows that the tangents at homologous points are parallel; and that the tangents at anti-homologous points meet on the radical axis. For these tangents are the limiting positions of homologous or anti-homologous chords. (1160)

1043 Let C, D be the inverse points of O with respect to two circles, centres A and B; then the constant product of anti-similitude

$$OP \,.\, OQ' \text{ or } OQ \,.\, OP' = OA \,.\, OD \text{ or } OB \,.\, OC.$$



PROOF.—By similar right-angled triangles,

$$OA : OT : OC \text{ and } OB : OR : OD;$$

therefore $OA \,.\, OD = OB \,.\, OC$ .... (1),

and also $OA \,.\, OC = OT^2 = OP \,.\, OQ,$ (III. 36)

and $OB \,.\, OD = OR^2 = OP' \,.\, OQ';$

therefore $OA \,.\, OB \,.\, OC \,.\, OD = OP \,.\, OQ \,.\, OP' \,.\, OQ',$

therefore &c., by (1).

1044 The foregoing definitions and properties (1037 to 1043), which have respect to the external centre of similitude O, hold good for the internal centre of similitude O', with the usual convention of positive and negative for distances measured from O' upon lines passing through it.

1045 Two circles will subtend equal angles at any point on the circumference of the circle whose diameter is OO', where O, O' are the centres of similitude (Fig. 1043). This circle is also coaxal with the given circles, and has been called the *circle of similitude.*

PROOF.—Let A, B be the centres, a, b the radii, and K any point on the circle, diameter OO'. Then, by (932),

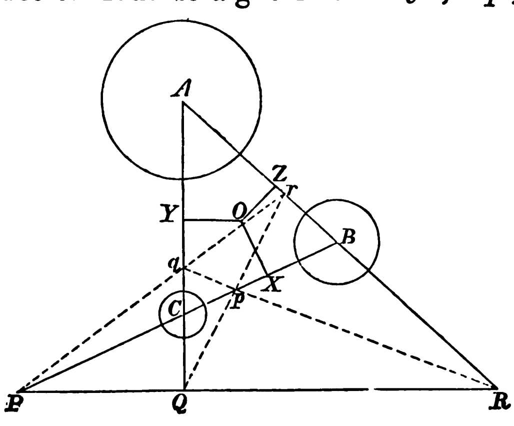
$$KA : KB = AO : BO = AO' : BO' = a : b,$$

by the definition (943);

therefore $a : KA = b : KB;$

that is, the sines of the halves of the angles in question are equal, which proves the first part. Also, because the tangents from K are in the constant ratio of the radii a, b, this circle is coaxal with the given ones, by (1036, 934).

1046 The six centres of similitude P, p, Q, q, R, r of three circles lie three and three on four straight lines PQR, Pqr, Qpr, Rpq, called *axes of similitude.*



PROOF. — Taking any three of the sets of points named, say P, q, r, they are shewn at once to be collinear by the transversal theorem (968) applied to the triangle ABC.

For the segments of its sides made by the points P, q, r are in the ratios of the radii of the circles.

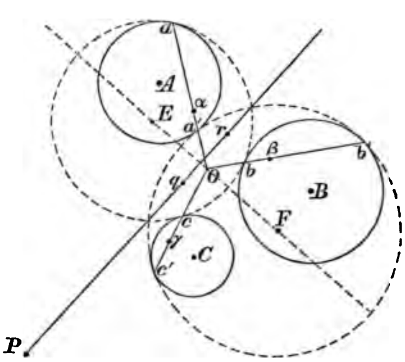
1047 From the investigation in (942), it appears that one circle touches two others in a pair of anti-homologous points, and that the following rule obtains:—

RULE.—*The right line joining the points of contact passes through the external or internal centre of similitude of the two circles according as the contacts are of the same or of different kinds.*

1048 DEFINITION.—Contact of curves is either *internal* or *external* according as the curvatures at the point of contact are in the same or opposite directions.

1049 *Gergonne's method of describing the circles which touch three given circles.*

Take Pqr, one of the four axes of similitude, and find its poles α, β, γ with respect to the given circles, centres A, B, C (1016). From O, the radical centre, draw lines through α, β, γ, cutting the circles in a, a', b, b', c, c'. Then a, b, c and a', b', c' will be the points of contact of two of the required circles.



PROOF.—*Analysis.*—Let the circles E, F touch the circles A, B, C in a, b, c, a', b', c'. Let $bc, b'c'$ meet in P; $ca, c'a'$ in q; and $ab, a'b'$ in r.

Regarding E and F as touched by A, B, C in turn, Rule (1047) shews that aa', bb', cc' meet in O, the centre of similitude of E and F; and (1041) shews that P, q, and r lie on the radical axis of E and F.

Regarding B and C, or C and A, or A and B, as touched by E and F in turn, Rule (1047) shews that P, q, r are the centres of similitude of B and C, C and A, A and B respectively; and (1041) shews that O is on the radical axis of each pair, and is therefore the radical centre of A, B, and C.

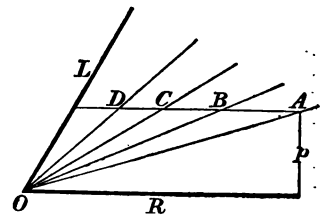
Again, because the tangents to E and F, at the anti-homologous points a, a', meet on Pqr, the radical axis of E and F (1042); therefore the point of meeting is the pole of aa' with respect to the circle A (1017). Therefore aa' passes through the pole of the line Pqr (1018). Similarly, bb' and cc' pass through the poles of the same line Pqr with respect to B and C. Hence the construction.

1050 In the given configuration of the circles A, B, C, the demonstration shews that each of the three internal axes of similitude Pqr, Qrp, Rpq (Fig. 1046) is a radical axis and common chord of two of the eight osculating circles which can be drawn. The external axis of similitude PQR is the radical axis of the two remaining circles which touch A, B, and C either all externally or all internally.

1051 The radical centre O of the three given circles is also the common internal centre of similitude of the four pairs of osculating circles. Therefore the central axis of each pair passes through O, and is perpendicular to the radical axis. Thus, in the figure, EF passes through O, and is perpendicular to Pqr.

Anharmonic Ratio.

1052 DEFINITION.—Let a pencil of four lines through a point O be cut by a transversal in the points A, B, C, D. The anharmonic ratio of the pencil is any one of the three fractions



$$\frac{AB.CD}{AD.BC} \text{ or } \frac{AB.CD}{AC.BD} \text{ or } \frac{AD.BC}{AC.BD}.$$

1053 The relation between these three different ratios is obtained from the equation

$$AB.CD+AD.BC = AC.BD.$$

Denoting the terms on the left side by p and q, the three anharmonic ratios may be expressed by

$$p:q, \quad p:p+q, \quad q:p+q.$$

The ratios are therefore mutually dependent. Hence, if the identity merely of the anharmonic ratio in any two systems is to be established, it is immaterial which of the three ratios is selected.

1054 In future, when the ratio of an anharmonic pencil $\{O, ABCD\}$ is mentioned, the form $AB.CD : AD.BC$ will be the one intended, whatever the actual order of the points A, B, C, D may be. For, it should be observed that, by making the line OD revolve about O, the ratio takes in turn each of the forms given above. This ratio is shortly expressed by the notation $\{O, ABCD\}$, or simply $\{ABCD\}$.

1055 If the transversal be drawn parallel to one of the lines, for instance OD, the two factors containing D become infinite, and their ratio becomes unity. They may therefore be omitted. The anharmonic ratio then reduces to $AB : BC$. Thus, when D is at infinity, we may write

$$\{O, ABC\infty\} = AB : BC.$$

1056 The anharmonic ratio

$$\frac{AB.CD}{AD.BC} = \frac{\sin AOB \sin COD}{\sin AOD \sin BOC},$$

and its value is therefore the same for all transversals of the pencil.

PROOF.—Draw OR parallel to the transversal, and let p be the perpendicular from A upon OR. Multiply each factor in the fraction by p. Then substitute $p.AB = OA.OB \sin AOB$, &c. (707).

1057 The anharmonic ratio (1056) becomes harmonic when its value is unity. See (933). The harmonic relation there defined may also be stated thus: four points divide a line harmonically when *the product of the extreme segments is equal to the product of the whole line and the middle segment.*

Homographic Systems of Points.

1058 DEFINITION.—If x, a, b, c be the distances of one variable point and three fixed points on a straight line from a point O on the same; and if x', a', b', c' be the distances of similar points on another line through O; then the variable points on the two lines will form two *homographic systems* when they are connected by the anharmonic relation

1059 $$\frac{(x-a)(b-c)}{(x-c)(a-b)} = \frac{(x'-a')(b'-c')}{(x'-c')(a'-b')}.$$

Expanding, and writing A, B, C, D for the constant coefficients, the equation becomes

1060 $$Axx'+Bx+Cx+D = 0.$$

From which

1061 $$x = -\frac{Cx'+D}{Ax'+B}, \quad \text{and} \quad x' = -\frac{Bx+D}{Ax+C}.$$

1062 THEOREM.—Any four arbitrary points x_1, x_2, x_3, x_4 on one of the lines will have four corresponding points x'_1, x'_2, x'_3, x'_4 on the other determined by the last equation, and *the two sets of points will have equal anharmonic ratios.*

PROOF.—This may be shown by actual substitution of the value of each x in terms of x', by (1061), in the harmonic ratio $\{x_1x_2x_3x_4\}$.

1063 If the distances of four points on a right line from a point O upon it, in order, are α, α', β, β', where α, β; α', β' are the respective roots of the two quadratic equations

$$ax^2+2hx+b = 0, \quad a'x^2+2h'x+b' = 0;$$

the condition that the two pairs of points may be *harmonically conjugate* is

1064 $$ab'+a'b = 2hh'.$$

PROOF.—The harmonic relation, by (1057), is

$$(\alpha-\alpha')(\beta-\beta') = (\alpha-\beta')(\alpha'-\beta).$$

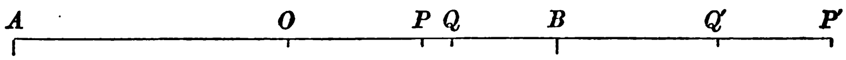
Multiply out, and substitute for the sums and products of the roots of the quadratics above in terms of their coefficients by (51, 52).

1065 If u_1, u_2 be the quadratic expressions in (1063) for two pairs of points, and if u represent a third pair harmonically conjugate with u_1 and u_2, then the pair of points u will also be harmonically conjugate with every pair given by the equation $u_1+\lambda u_2 = 0$, where λ is any constant. For the condition (1064) applied to the last equation will be identically satisfied.

Involution.

1066 DEFINITIONS.—Pairs of *inverse points* PP', QQ', &c., on the same right line, form a system *in involution*, and the relation between them, by (1000), is

$$OP\,.\,OP' = OQ\,.\,OQ' = \&\text{c.} = k^2.$$



The radius of the circle of inversion is k, and the centre O is called the *centre of the system*. Inverse points are also termed *conjugate points*.

When two inverse points coincide, the point is called a *focus*.

1067 The equation $OP^2 = k^2$ shows that there are two foci A, B at the distance k from the centre, and on opposite sides of it, real or imaginary according as any two inverse points lie on the same side or on opposite sides of the centre.

1068 If the two homographic systems of points in (1058) be on the *same line*, they will constitute a system *in involution* when $B = C$.

PROOF.—Equation (1060) may now be written

$$Axx'+H(x+x')+B = 0,$$

or
$$\left(x+\frac{H}{A}\right)\left(x'+\frac{H}{A}\right) = \frac{H^2}{A^2}-\frac{B}{A} = k^2,$$

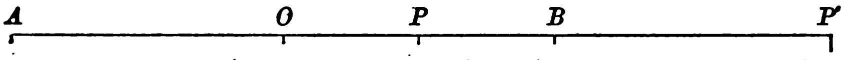
a constant. Therefore $-\frac{H}{A}$ is the distance of the origin O from the centre of inversion. Measuring from this centre, the equation becomes $\xi\xi' = k^2$, representing a system in involution.

1069 *Any four points whatever* of a system in involution on a right line have their anharmonic ratio equal to that of their four conjugates.

PROOF.—Let p, p'; q, q'; r, r'; s, s' be the distances of the pairs of inverse points from the centre.

In the anharmonic ratio of *any* four of the points, for instance $\{pq'rs\}$, substitute $p = k^2 \div p'$, $q' = k^2 \div q$, &c., and the result is the anharmonic ratio $\{p'qr's'\}$.

1070 Any two inverse points P, P' are in harmonic relation with the foci A, B.



PROOF.—Let p, p' be the distances of P, P' from the centre O; then

$pp' = k^2$, therefore $\frac{p'}{k} = \frac{k}{p}$, therefore $\frac{p'+k}{p'-k} = \frac{k+p}{k-p}$;

that is, $$\frac{AP'}{BP'} = \frac{AP}{BP}. \qquad (933)$$

1071 If a system of points in involution be given, as in (1068), by the equation

$$Axx' + H(x+x') + B = 0 \;\ldots\ldots\ldots\ldots\ldots\; (1);$$

and a pair of conjugate points by the equation

$$ax^2 + 2hx + b = 0 \;\ldots\ldots\ldots\ldots\ldots\; (2);$$

the necessary relation between a, h, and b is

1072 $$Ab + Ba = 2Hh.$$

PROOF.—The roots of equation (2) must be simultaneous values of x, x' in (1); therefore substitute in (1)

$$x+x' = -\frac{2h}{a} \quad \text{and} \quad xx' = \frac{b}{a}. \qquad (51)$$

1073 COR.—A system in involution may be determined from two given pairs of corresponding points.

Let the equations for these points be

$$ax^2 + 2hx + b = 0 \quad \text{and} \quad a'x^2 + 2hx + b' = 0.$$

Then there are two conditions (1072),

$$Ab + Ba = 2Hh \quad \text{and} \quad Ab' + Ba' = 2Hh',$$

from which A, H, B can be found.

A geometrical solution is given in (985). C, D; C', D' are, in that construction, pairs of inverse points, and I is the centre of a system in involution defined by a series of coaxal circles (1022). Each circle intersects the central axis in a pair of inverse points with respect to the circle whose centre is O and radius δ.

1074 The relations which have been established for a system of *collinear* points may be transferred to a system of concurrent lines by the method of (1056), in which the distance between two points corresponds to the sine of the angle between two lines passing through those points.

The Method of Projection.

1075 Definitions.—The *projection* of any point P in space (Fig. of 1079) is the point p in which a right line OP, drawn from a fixed point O called the *vertex*, intersects a fixed plane called the *plane of projection.*

If all the points of any figure, plane or solid, be thus projected, the figure obtained is called the *projection* of the original figure.

1076 *Projective Properties.*—The projection of a right line is a right line. The projections of parallel lines are parallel. The projections of a curve, and of the tangent at any point of it, are another curve and the tangent at the corresponding point.

1077 The anharmonic ratio of the segments of a right line is not altered by projection; for the line and its projection are but two transversals of the same anharmonic pencil. (1056)

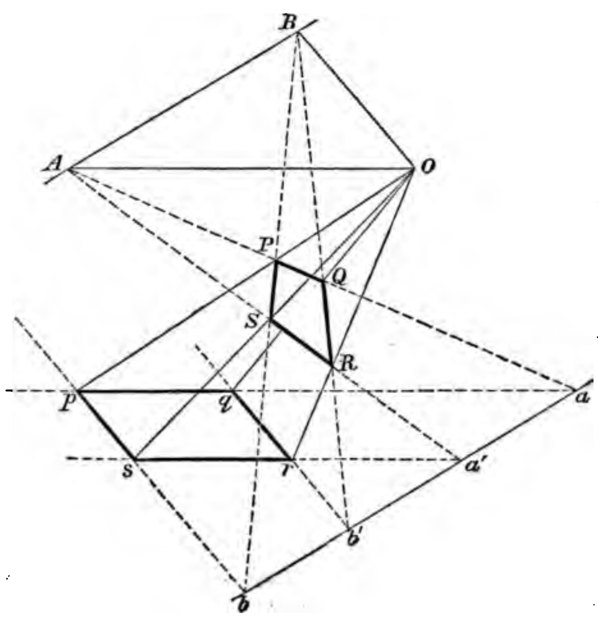
1078 Also, any relation between the segments of a line similar to that in (1015), in which each letter occurs in every term, is a *projective property.* [Proof as in (1056).

1079 *Theorem.*—Any quadrilateral $PQRS$ may be projected into a parallelogram.

Construction. — Produce PQ, SR to meet in A, and PS, QR to meet in B.

Then, with any point O for vertex, project the quadrilateral upon any plane pab parallel to OAB. The projected figure $pqrs$ will be a parallelogram.

Proof. — The planes OPQ, ORS intersect in OA, and they intersect the plane of projection which is parallel to OA in the lines pq, rs. Therefore pq and rs are parallel to OA, and therefore to each other. Similarly, ps, qr are parallel to OB.



1080 Cor. 1.—The opposite sides of the parallelogram *pqrs* meet in two points at infinity, which are the projections of the points *A*, *B*; and *AB* itself, which is the third diagonal of the complete quadrilateral *PQRS*, is projected into a line at infinity.

1081 Hence, to project any figure so that a certain line in it may pass to infinity—*Take the plane of projection parallel to the plane which contains the given line and the vertex.*

1082 Cor. 2.—To make the projection of the quadrilateral a rectangle, it is only necessary to make *AOB* a right angle.

On Perspective Drawing.

1083 Taking the parallelogram *pqrs*, in (1079), for the original figure, the quadrilateral *PQRS* is its projection on the plane *ABab*. Suppose this plane to be the plane of the paper. Let the planes *OAB*, *pab*, while remaining parallel to each other, be turned respectively about the fixed parallel lines *AB*, *ab*. In every position of the planes, the lines *Op*, *Oq*, *Or*, *Os* will intersect the dotted lines in the same points *P*, *Q*, *R*, *S*. When the planes coincide with that of the paper, *pqrs* becomes a *ground plan* of the parallelogram, and *PQRS* is the representation of it in perspective.

AB is then called the *horizontal* line, *ab* the *picture line*, and the plane of both the *picture plane*.

1084 To find the projection of any point *p* in the *ground plan.*

Rule.—*Draw* pb *to any point* b *in the picture line, and draw* OB *parallel to* pb, *to meet the horizontal line in* B. *Join* Op, Bb, *and they will intersect in* P, *the point required.*

In practice, *pb* is drawn perpendicular to *ab*, and *OB* therefore perpendicular to *AB*. The point *B* is then called the *point of sight*, or *centre of vision*, and *O* the *station point*.

1085 To find the projection of a point in the *ground plan*, not in the original plane, but at a perpendicular distance *c* above it.

Rule.—*Take a new picture line parallel to the former, and at a distance above it* $= c \operatorname{cosec} a$, *where* a *is the angle between the original plane and the plane of projection.* For a plane through the given point, parallel to the original plane, will intersect the plane of projection in the new *picture line* so constructed.

Thus, every point of a figure in the *ground plan* is transferred to the drawing.

1086 The whole theory of perspective drawing is virtually included in the foregoing propositions. The original plane is commonly horizontal, and the plane of projection vertical. In this case, $\operatorname{cosec} a = 1$, and the height of the *picture line* for any point is equal to the height of the point itself above the original plane.

The distance *BO*, when *B* is the *point of sight*, may be measured along *AB*, and *bp* along *ab*, in the opposite direction; for the line *Bb* will continue to intersect *Op* in the point *P*.

Orthogonal Projection.

1087 Definition.—In orthogonal projection the lines of projection are parallel to each other, and perpendicular to the plane of projection. The vertex in this case may be considered to be at infinity.

1088 The projections of parallel lines are parallel, and the projected segments are in a constant ratio to the original segments.

1089 Areas are in a constant ratio to their projections.

For, lines parallel to the intersection of the original plane and the plane of projection are unaltered in length, and lines at right angles to the former are altered in a constant ratio. This ratio is the ratio of the areas, and is the cosine of the angle between the two planes.

Projections of the Sphere.

1090 In *Stereographic projection*, the vertex is on the surface of the sphere, and the diameter through the vertex is perpendicular to the plane of projection which passes through the other extremity of the diameter. The projection is therefore the *inversion* of the surface of the sphere (1012), and the diameter is the constant k.

1091 In *Globular projection*, the vertex is taken at a distance from the sphere equal to the radius $\div\sqrt{2}$, and the diameter through the vertex is perpendicular to the plane of projection.

1092 In *Gnomonic projection*, which is used in the construction of sun-dials, the vertex is at the centre of the sphere.

1093 *Mercator's projection*, which is employed in navigation, and sometimes in maps of the world, is not a projection at all as defined in (1075). Meridian circles of the sphere are represented on a plane by parallel right lines at intervals equal to the intervals on the equator. The parallels of latitude are represented by right lines perpendicular to the meridians, and at increasing intervals, so as to preserve the actual ratio between the increments of longitude and latitude at every point.

With r for the radius of the sphere, the distance, on the chart, from the equator of a point whose latitude is λ, is $= r \log \tan (45^\circ + \frac{1}{2}\lambda)$.

Additional Theorems.

1094 The sum of the squares of the distances of any point P from n equidistant points on a circle whose centre is O and radius r $= n(r^2 + OP^2)$.

PROOF.—Sum the values of PB^2, PC^2, &c., given in (819), and apply (803). This theorem is the generalization of (923).

1095 In the same figure, if P be on the circle, the sum of the squares of the perpendiculars from P on the radii OB, OC, &c., is equal to $\frac{1}{2}nr^2$.

PROOF.—Describe a circle upon the radius through P as diameter, and apply the foregoing theorem to this circle.

1096 COR. 1.—The sum of the squares of the intercepts on the radii between the perpendiculars and the centre is also equal to $\frac{1}{2}nr^2$. (I. 47)

1097 COR. 2.—The sum of the squares of the perpendiculars from the equidistant points on the circle to any right line passing through the centre is also equal to $\frac{1}{2}nr^2$.

Because the perpendiculars from two points on a circle to the diameters drawn through the points are equal.

1098 COR. 3.—The sum of the squares of the intercepts on the same right line between the centre of the circle and the perpendiculars is also equal to $\frac{1}{2}nr^2$. (I. 47)

If the radii of the inscribed and circumscribed circles of a regular polygon of n sides be r, R, and the centre O; then,

1099 I. The sum of the perpendiculars from any point P upon the sides is equal to nr.

1100 II. If p be the perpendicular from O upon any right line, the sum of the perpendiculars from the vertices upon the same line is equal to np.

1101 III. The sum of the squares of the perpendiculars from P on the sides is $= n(r^2 + \frac{1}{2}OP^2)$.

1102 IV. The sum of the squares of the perpendiculars from the vertices upon the right line is $= n(p^2 + \frac{1}{2}R^2)$.

PROOF.—In theorem I., the values of the perpendiculars are given by $r - OP\cos\left(\theta + \frac{2m\pi}{n}\right)$, with successive integers for m. Add together the n values, and apply (803).

Similarly, to prove II.; take for the perpendiculars the values

$$p - R\cos\left(\theta + \frac{2m\pi}{n}\right).$$

To prove III. and IV., take the same expressions for the perpendiculars; square each value; add the results, and apply (803, 804).

For additional propositions in the subjects of this section, see the section entitled *Plane Coordinate Geometry.*

GEOMETRICAL CONICS.

THE SECTIONS OF THE CONE.

1150 DEFINITIONS.—A *Conic Section* or *Conic* is the curve AP in which any plane intersects the surface of a right cone.

A *right cone* is the solid generated by the revolution of one straight line about another which it intersects in a fixed point at a constant angle.

Let the axis of the cone, in Fig. (1) or Fig. (2), be in the plane of the paper, and let the cutting plane $PMXN$ be perpendicular to the paper. (*Read either the accented or unaccented letters throughout.*) Let a sphere be inscribed in the cone, touching it in the circle EQF and touching the cutting plane in the point S, and let the cutting plane and the plane of the circle EQF intersect in XM. The following theorem may be regarded as the *defining property* of the curve of section.

1151 *Theorem.*—The distance of any point P on the conic from the point S, called the *focus*, is in a constant ratio to PM, its distance from the line XM, called the *directrix*, or

$$PS : PM = PS' : PM' = AS : AX = e, \text{ the } \textit{eccentricity}.$$

[*See next page for the Proof.*]

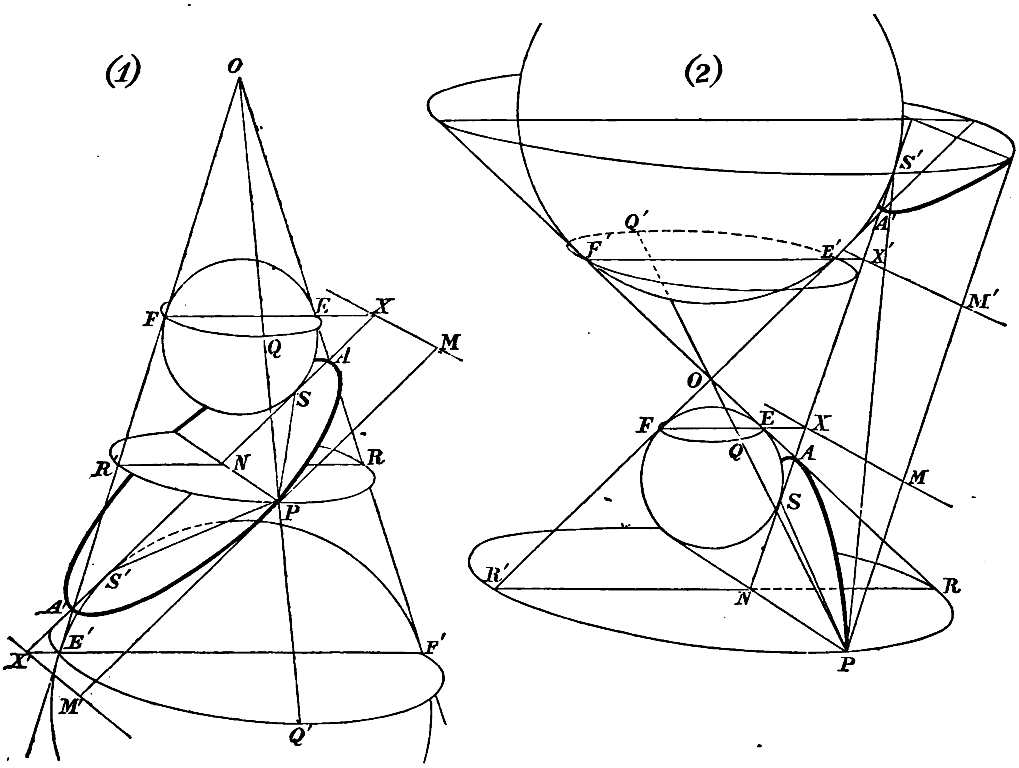
1152 COR.—The conic may be generated in a plane from either focus S, S', and either directrix XM, $X'M'$, by the law just proved.

1153 The conic is an *Ellipse*, a *Parabola*, or an *Hyperbola*, according as e is less than, equal to, or greater than unity. That is, according as the cutting plane emerges on both sides of the lower cone, or is parallel to a side of the cone, or intersects both the upper and lower cones.

1154 All sections made by parallel planes are similar; for the inclination of the cutting plane determines the ratio $AE : AX$.

1155 The limiting forms of the curve are respectively—a circle when e vanishes, and two coincident right lines when e becomes infinite.

PROOF OF THEOREM 1151.—Join P, S and P, O, cutting the circular section in Q, and draw PM parallel to NX. Because all tangents from the same point, O or A, to either sphere are equal, therefore $RE = PQ = PS$ and $AE = AS$. Now, by (VI. 2), $RE : NX = AE : AX$ and $NX = PM$; therefore $PS : PM = AS : AX$, a constant ratio denoted by e and called the *eccentricity* of the conic.



Referring the letters either to the ellipse or the hyperbola in the subjoined figure, let C be the middle point of AA' and N any other point on it. Let DD', RR' be the two circular sections of the cone whose planes pass through C and N; BCB' and PN the intersections with the plane of the conic. In the ellipse, BC is the common ordinate of the ellipse and circle; but, in the hyperbola, BC is to be taken equal to the tangent from C to the circle DD'.

1156 The fundamental equation of the ellipse or hyperbola is

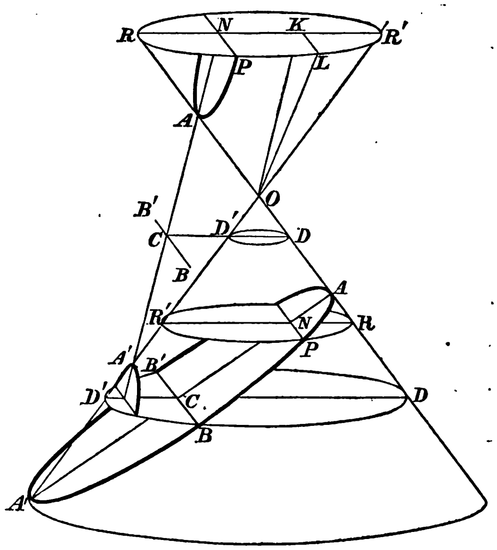
$$PN^2 : AN.NA' = BC^2 : AC^2.$$

PROOF.—$PN^2 = NR.NR'$ and $BC^2 = CD.CD'$ (III. 35, 36). Also, by similar triangles (VI. 2, 6), $NR : CD = AN : AC$ and $NR' : CD' = A'N : A'C$. Multiply the last equations together.

1157 Cor. 1.—*PN* has equal values at two points equi-distant from *AA'*. Hence the curve is symmetrical with respect to *AA'* and *BB'*.

These two lines are called the *major* and *minor axes*, otherwise the *transverse* and *conjugate* axes of the conic.

When the axes are equal, or $BC = AC$, the ellipse becomes a circle, and the hyperbola becomes *rectangular* or *equilateral*.



1158 Any ellipse or hyperbola is the orthogonal projection of a circle or rectangular hyperbola respectively.

Proof.—Along the ordinate *NP*, measure $NP' = AN \, . \, NA'$; therefore by the theorem $PN : P'N = BC : AC$. Therefore a circle or rectangular hyperbola, having *AA'* for one axis, and having its plane inclined to that of the conic at an angle whose cosine $= BC \div AC$, projects orthogonally into the ellipse or hyperbola in question, by (1089). See Note to (1201).

1159 Hence any *projective property* (1076–78), which is known to belong to the circle or rectangular hyperbola, will also be universally true for the ellipse and hyperbola respectively.

THE ELLIPSE AND HYPERBOLA.

Joint properties of the Ellipse and Hyperbola.

1160 Definitions.—The *tangent* to a curve at a point *P* (Fig. 1166) is the right line *PQ*, drawn through an adjacent point *Q*, in its ultimate position when *Q* is made to coincide with *P*.

The *normal* is the perpendicular to the tangent through the point of contact.

In (Fig. 1171), referred to rectangular axes through the centre C (see Coordinate Geometry); the length CN is called the *abscissa; PN* the *ordinate; PT* the *tangent; PG* the *normal; NT* the *subtangent;* and NG the *subnormal.* S, S' are the *foci; XM, X'M'* the *directrices; PS, PS'* the *focal distances*, and a double ordinate through S the *Latus Rectum.*

The *auxiliary circle* (Fig. 1173) is described upon AA' as diameter.

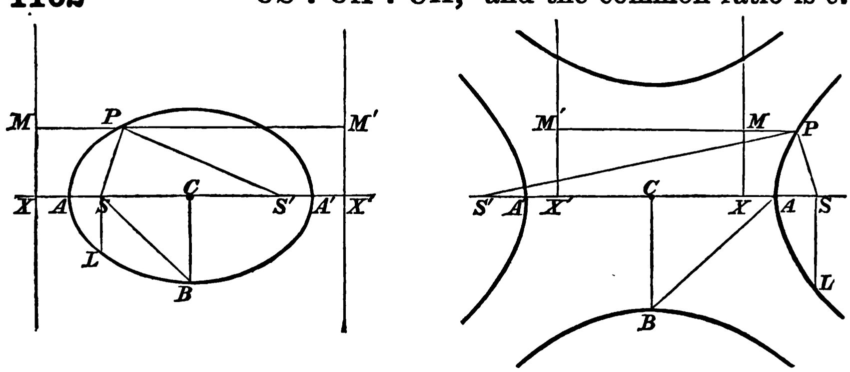
A diameter parallel to the tangent at the extremity of another diameter is termed a *conjugate diameter* with respect to the other.

The *conjugate hyperbola* has BC for its major, and AC for its minor axis (1157).

1161 The following theorems (1162) to (1181) are deduced from the property $PS : PM = e$ obtained in (1151).

The propositions and demonstrations are nearly identical for the ellipse and the hyperbola, any difference in the application being specified..

1162 $CS : CA : CX$, and the common ratio is e.



PROOF.—By (1151), $e = \dfrac{AS}{AX} = \dfrac{A'S}{A'X} = \dfrac{\frac{1}{2}(A'S \pm AS)}{\frac{1}{2}(A'X \pm AX)} = \dfrac{CS}{CA}$ or $\dfrac{CA}{CX}$.

1163 In the ellipse the sum, and in the hyperbola the difference, of the focal distances of P is equal to the major axis,

or $$PS' \pm PS = AA'.$$

PROOF.—With the same figures we have, in the ellipse, by (1151),

$e = \dfrac{PS+PS'}{PM+PM'} = \dfrac{PS+PS'}{XX'}$, and also $e = \dfrac{AS+A'S}{AX+A'X} = \dfrac{AA'}{XX'}$, therefore &c.

For the hyperbola take *difference* instead of *sum.*

1164 $CS^2 = AC^2 - BC^2$ in the ellipse.

[For $BS = AC$, by (1163).

$CS^2 = AC^2 + BC^2$ in the hyperbola.

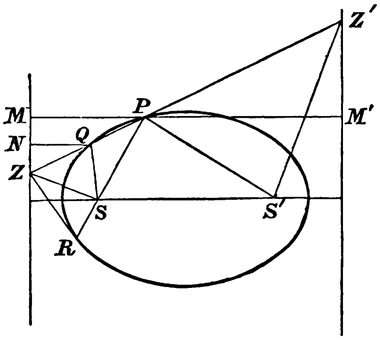
[By assuming BC. See (1176).

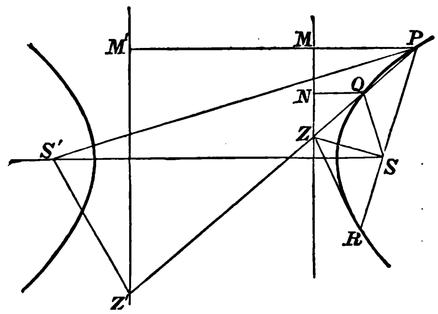
1165 $BC^2 = SL \,.\, AC.$

PROOF.—(Figs. of 1162) $SL : SX = CS : CA$, (1151, 1162)

$\therefore SL.AC = CS.SX = CS(CX \sim CS) = CA^2 \sim CS^2$ (1162) $= BC^2$ (1164).

1166 If a right line through P, Q, two points on the conic, meets the directrix in Z, then SZ bisects the angle QSR.





PROOF.—By similar triangles, $ZP : ZQ = MP : NQ = SP : SQ$ (1151), therefore by (VI. A.)

1167 If PZ be a tangent at P, then PSZ and $PS'Z'$ are right angles.

PROOF.—Make Q coincide with P in the last theorem.

1168 The tangent makes equal angles with the focal distances.

PROOF.—In (1166), $PS : PS' = PM : PM'$ (1151) $= PZ : PZ'$; therefore, when PQ becomes the tangent at P, $\angle SPZ = S'PZ'$, by (1167) and (VI. 7).

1169 The tangents at the extremities of a focal chord intersect in the directrix.

PROOF.—(Figs. of 1166). Join ZR; then, if ZP is a tangent, ZR is also, for (1167) proves RSZ to be a right angle.

1170 $CN \,.\, CT = AC^2.$

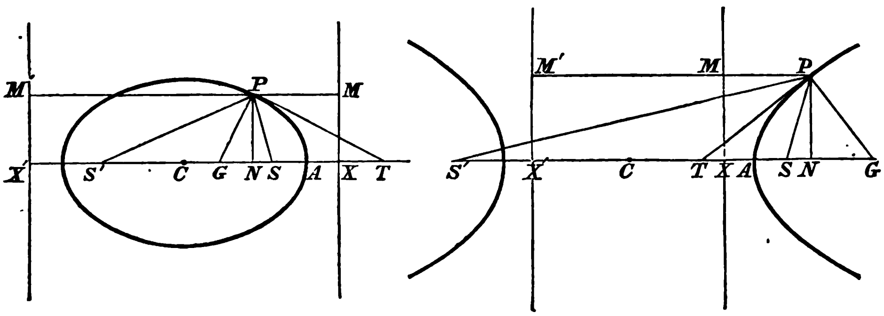
PROOF.—(Figs. 1171.) $\dfrac{TS'}{TS} = \dfrac{PS'}{PS}$ (VI. 3, A.) $= \dfrac{PM'}{PM}$ (1151) $= \dfrac{NX'}{NX}$,

therefore $\dfrac{TS'+TS}{TS'-TS} = \dfrac{NX'+NX}{NX'-NX}$, or $\dfrac{2CT}{2CS} = \dfrac{2CX}{2CN}$,

therefore $CN.CT = CS.CX = AC^2.$ (1162)

1171 If PG be the normal,

$$GS : PS = GS' : PS' = e.$$



PROOF.—By (1168) and (VI. 3, A.),

$$\frac{GS}{PS} = \frac{GS'}{PS'} = \frac{GS'+GS}{PS'+PS} = \frac{2CS}{2CA} = e. \qquad (1162)$$

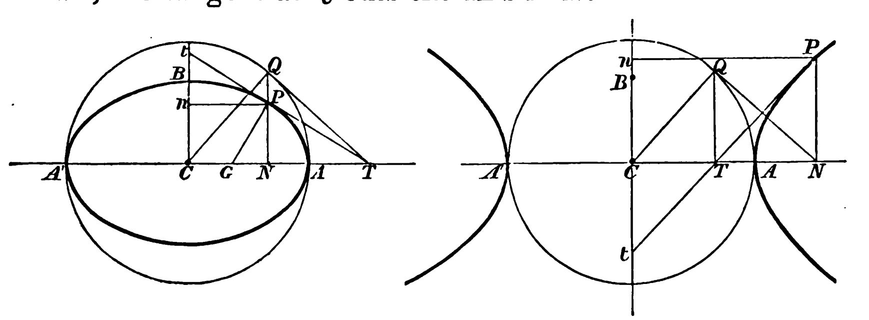
But, for the hyperbola, change *plus* to *minus*.

1172 The subnormal and the abscissa are as the squares of the axes, or $NG : NC = BC^2 : AC^2$.

PROOF.—(Figs. 1171.) Exactly as in (1170), taking the normal instead of the tangent, we obtain $\frac{CG}{CS} = \frac{CN}{CX}$, $\therefore \frac{CN}{CG} = \frac{CX}{CS} = \frac{CA^2}{CS^2}$ (1162),

$$\therefore \frac{CN \sim CG}{CN} = \frac{CA^2 \sim CS^2}{CA^2}, \text{ or } \frac{NG}{NC} = \frac{BC^2}{AC^2} \ (1164).$$

1173 The tangents at P and Q, the corresponding points on the ellipse and auxiliary circle, meet the axis in the same point T. But in the hyperbola, the ordinate TQ of the circle being drawn, the tangent at Q cuts the axis in N.



PROOF.—For the ellipse: Join TQ. Then $CN.CT = CQ^2$ (1170); therefore CQT is a right angle (VI. 8); therefore QT is a tangent.

For the hyperbola: Interchange N and T.

1174 $PN : QN = BC : AC.$

PROOF.—(Figs. 1173). $NG.NT = PN^2$, and $CN.NT = QN^2$. (VI. 8) Therefore $NG : NC = PN^2 : QN^2$; therefore, by (1172).

This proposition is equivalent to (1158), and shows that an ellipse is the orthogonal projection of a circle equal to the auxiliary circle.

1175 COR.—The area of the ellipse is to that of the auxiliary circle as $BC : AC$ (1089).

1176 $PN^2 : AN.NA' = BC^2 : AC^2.$

PROOF.—By (1174), since $QN^2 = AN.NA'$ (III. 35, 36). An independent proof of this theorem is given in (1156). The construction for BC in the hyperbola in (1164) is thus verified.

1177 $Cn.Ct = BC^2.$

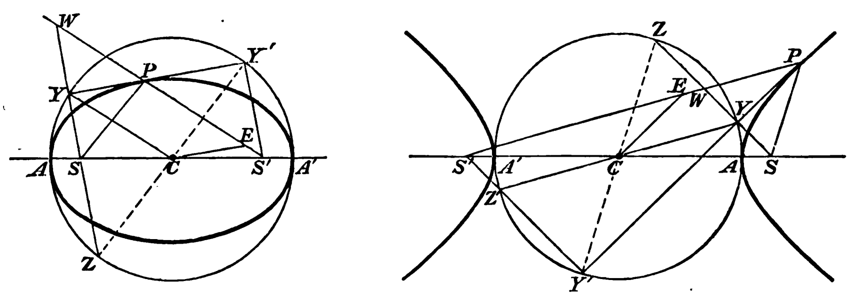
PROOF.—(Figs. 1173.)

$$\frac{Ct}{CT} = \frac{PN}{NT};\ \therefore \frac{Cn.Ct}{CN.CT} = \frac{PN^2}{CN.NT} = \frac{PN^2}{QN^2}\ \text{(VI. 8)} = \frac{PN^2}{AN.NA'}\ \text{(III. 35, 36)}.$$

Therefore, by (1170) and (1176), $Cn.Ct : AC^2 = BC^2 : AC^2$.

1178 If SY, $S'Y'$ are the perpendiculars on the tangent, then Y, Y' are points on the auxiliary circle, and

$$SY.S'Y' = BC^2.$$



PROOF.—Let PS meet SY in W. Then $PS = PW$ (1168). Therefore $S'W = AA'$ (1163). Also, $SY = YW$, and $SC = CS'$. Therefore $CY = \frac{1}{2}S'W = AC$. Similarly $CY' = AC$. Therefore Y, Y' are on the circle.

Hence ZY' is a diameter (III. 31), and therefore $SZ = S'Y'$, by similar triangles; therefore $SY.SZ = SA.SA'$ (III. 35, 36) $= CS^2 \sim CA^2$ (II. 5) $= BC^2$ (1164).

1179 COR.—If CE be drawn parallel to the tangent at P, then $PE = CY = AC.$

1180 PROBLEM.—To draw tangents from any point O to an ellipse or hyperbola.

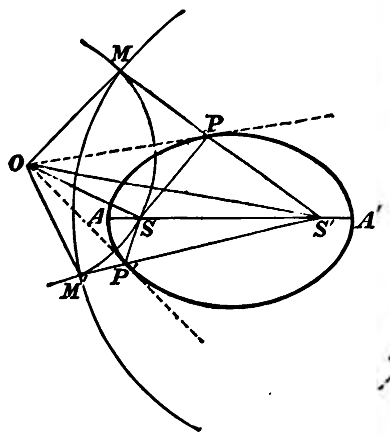
CONSTRUCTION.—(Figs. 1181.) Describe two circles, one with centre O and radius OS, and another with centre S' and radius $= AA'$, intersecting in M,

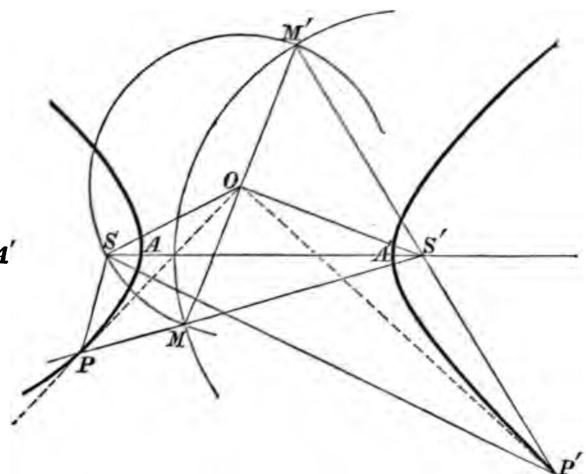
M'. Join MS', $M'S'$. These lines will intersect the curve in P, P', the points of contact. For another method see (1204).

PROOF.—By (1163), $PS' \pm PS = AA' = S'M$ by construction. Therefore $PS = PM$, therefore $\angle OPS = OPM$ (I. 8), therefore OP is a tangent by (1168).

Similarly $P'S = P'M'$, and OP' is a tangent.

1181 The tangents OP, OP' subtend equal angles at either focus.



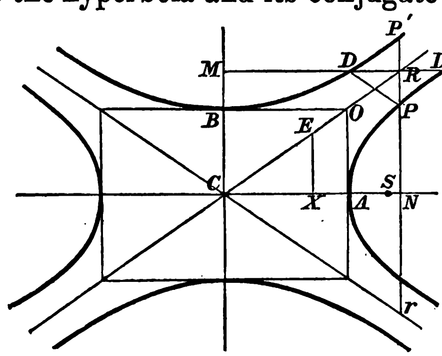


PROOF.—The angles OSP, OSP' are respectively equal to OMP, $OM'P'$, by (I. 8), as above; and these last angles are equal, by the triangles $OS'M$, $OS'M'$, and (I. 8). Similarly at the other focus.

Asymptotic Properties of the Hyperbola.

1182 DEF. — The *asymptotes* of the hyperbola are the diagonals of the rectangle formed by tangents at the vertices A, A', B, B'.

1183 If the ordinates RN, RM from any point R on an asymptote cut the hyperbola and its conjugate in P, P', D, D',



then either of the following pairs of equations will define both the branches of each curve—

$$RN^2 - PN^2 = BC^2 = P'N^2 - RN^2 \quad \ldots\ldots\ldots\ldots (1),$$
$$RM^2 - DM^2 = AC^2 = D'M^2 - RM^2 \quad \ldots\ldots\ldots\ldots (2).$$

PROOF.—Firstly, to prove (1): By proportion from the similar triangles RNC, OAC, we have $\frac{RN^2}{CN^2} = \frac{BC^2}{AC^2} = \frac{PN^2}{CN^2 - AC^2}$;

by (1176), since $AN.NA' = CN^2 - AC^2$. By (II. 6)

Therefore $\frac{RN^2 - PN^2}{AC^2} = \frac{BC^2}{AC^2}$, by the theorem (69);

therefore $RN^2 - PN^2 = BC^2$.

Also, by (1176), applied to the conjugate hyperbola, the axes being now reversed, $\frac{CN^2}{P'N^2 - BC^2} = \frac{AC^2}{BC^2} = \frac{CN^2}{RN^2}$, by similar triangles;

therefore $P'N^2 - BC^2 = RN^2$ or $P'N^2 - RN^2 = BC^2$.

Secondly, to prove (2): By proportion from the triangles RMC, OBC, we have $\frac{RM^2}{CM^2} = \frac{AC^2}{BC^2} = \frac{DM^2}{CM^2 - BC^2}$,

by (1176), applied to the conjugate hyperbola, for in this case we should have $BM.MB' = CM^2 - BC^2$.

Therefore $\frac{RM^2 - DM^2}{BC^2} = \frac{AC^2}{BC^2}$; therefore $RM^2 - DM^2 = AC^2$.

Also, by (1176), since CM, $D'M$ are equal to the coordinates of D',

$$\frac{CM^2}{D'M^2 - AC^2} = \frac{BC^2}{AC^2} = \frac{CM^2}{RM^2}, \text{ by similar triangles;}$$

therefore $D'M^2 - AC^2 = RM^2$ or $D'M^2 - RM^2 = AC^2$.

1184 COR. 1.—If the same ordinates RN, RM meet the other asymptote in r and r', then

$$PR.Pr = BC^2 \text{ and } DR.Dr' = AC^2. \quad \text{(II. 5)}$$

1185 COR. 2.—As R recedes from C, PR and DR continually diminish. Hence the curves continually approach the asymptote.

1186 If XE be the directrix, $CE = AC$.

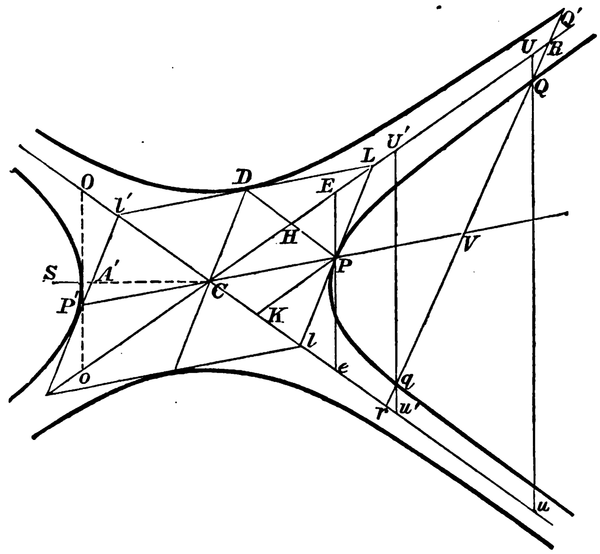
PROOF.— $CE : CO = CX : CA = CA : CS$ and $CS = CO$. (1164)

1187 PD is parallel to the asymptote.

PROOF.— $\frac{RN^2}{RM^2} = \frac{BC^2}{AC^2} = \frac{RN^2 - PN^2}{RM^2 - DM^2}$ (1183) $= \frac{PN^2}{DM^2}$ (69).

Therefore $RN : PN = RM : DM$; therefore, by (VI. 2).

1188 The segments of any right line between the curve and the asymptote are equal, or $QR = qr$.



PROOF.— $QR : QU = qR : qU'$ } Compound the ratios,
and $Qr : Qu = qr : qu'$ } and employ (1184).

1189 COR. 1.—$PL = Pl$ and $QV = qV$.

1190 COR. 2.—$CH = HL$. Because PD is parallel to lC. (1187)

1191 $QR.Qr = PL^2 = RV^2 - QV^2 = Q'V^2 - RV^2$.

PROOF.— $QR : QU = PL : PE$ } Compound the ratios. Therefore, by (1184),
and $Qr : Qu = Pl : Pe$ } $QR.Qr = PL.Pl = PL^2$ (1189).

1192 $4PH.PK = CS^2$.

PROOF.— $PH : PE = CO : Oo$ } $\therefore PH.PK : PE.Pe = CO^2 : Oo^2$
and $PK : Pe = Co : Oo$ } $= CS^2 : 4BC^2$; therefore, by (1184).

Joint Properties of the Ellipse and Hyperbola resumed.

If PCP' be a diameter, and QV an ordinate parallel to the conjugate diameter CD (Figs. 1195 and 1188).

1193 $QV^2 : PV.VP' = CD^2 : CP^2$.

This is the fundamental equation of the conic, equation (1176) being the most important form of it.

Otherwise:

In the ellipse, $QV^2 : CP^2 - CV^2 = CD^2 : CP^2$.

In the hyperbola, $QV^2 : CV^2 - CP^2 = CD^2 : CP^2$;

and $Q'V^2 : CV^2 + CP^2 = CD^2 : CP^2$.

PROOF.—(*Ellipse.* Fig. 1195.)—By orthogonal projection from a circle. If C, P, P', D, Q, V are the projections of c, p, p', d, q, v on the circle; $qv^2 = pv \cdot vp'$ and $cd^2 = cp^2$. The proportion is therefore true in the case of the circle. Therefore &c., by (1088).

(*Hyperbola.* Fig. 1188)—

$$\frac{CD^2}{CP^2} = \frac{RV^2}{CV^2} = \frac{PL^2}{CP^2} = \frac{RV^2 \pm PL^2}{CV^2 \pm CP^2} = \frac{QV^2}{CV^2 - CP^2} \text{ or } \frac{Q'V^2}{CV^2 + CP^2}. \quad (1191)$$

1194 The parallelogram formed by tangents at the extremities of conjugate diameters is of constant area, and therefore, PF being perpendicular to CD (Figs. 1195),

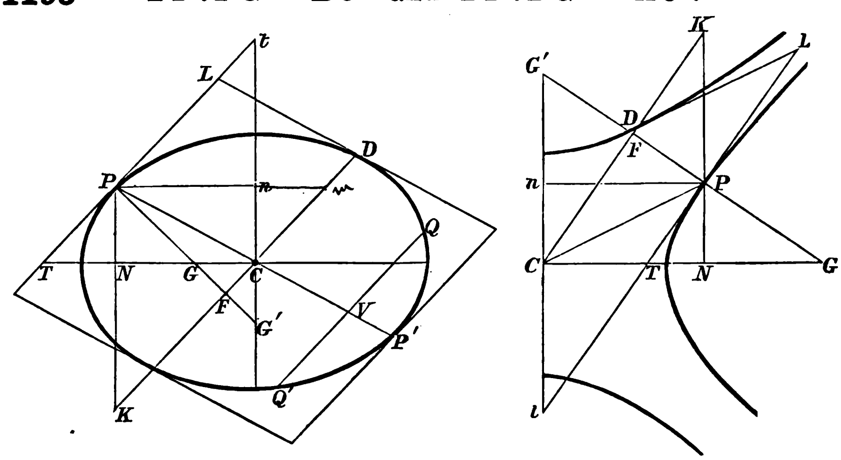
$$PF \cdot CD = AC \cdot BC.$$

PROOF.—(*Ellipse.*)—By orthogonal projection from the circle (1089).

(*Hyperbola.* Fig. 1188.)—$CL \cdot Cl = 4PH \cdot PK = CO \cdot Co$ (1192); therefore, by (VI. 15), $\Delta LCl = OCo = AC \cdot BC$.

If PF intersects the axes in G and G',

1195 $PF \cdot PG = BC^2$ and $PF \cdot PG' = AC^2$.



PROOF.— $PF \cdot PG = PK \cdot PN = Cn \cdot Ct = BC^2$ (1177). Similarly for $PF \cdot PG'$.

1197 COR.— $PG \cdot PG' = CD^2 = PT \cdot Pt$. By (1194)

1198 The diameter bisects all chords parallel to the tangent at its extremity.

PROOF.—(*Ellipse.* Fig. 1195.)—By projection from the circle (1088) $QV = VQ'$. (*Hyperbola.*) By (1189.)

1199 COR. 1.—The tangents at the extremities of any chord meet on the diameter which bisects it.

PROOF.—The secants drawn through the extremities of two parallel chords meet on the diameter which bisects them (VI. 4), and the tangents are the limiting positions of the secants when the parallel chords coincide.

1200 COR. 2.—If the tangents from a point are equal, the diameter through the point must be a principal axis. (I. 8)

1201 COR. 3.—The chords joining any point Q on the curve with the extremities of a diameter PP', are parallel to conjugate diameters, and are called *supplemental chords.*

For the diameter bisecting PQ is parallel to $P'Q$ (VI. 2). Similarly the diameter bisecting $P'Q$ is parallel to PQ.

1202 Diameters are mutually conjugate; If CD be parallel to the tangent at P, CP will be parallel to the tangent at D.

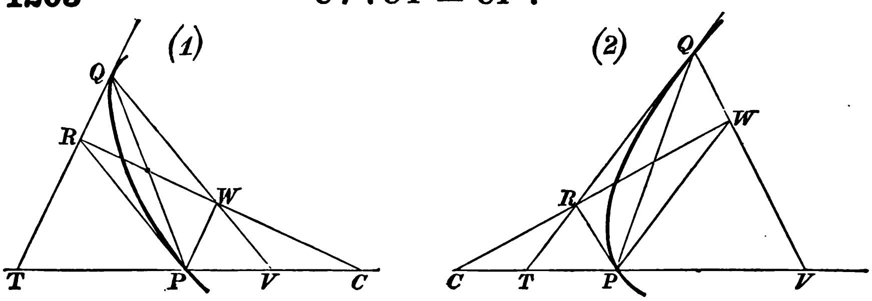
PROOF.—(*Ellipse.* Fig. 1205.)—By projection from the circle (1088).

NOTE.—Observe that, if the ellipse in the figure with its ordinates and tangents be turned about the axis Tt through the angle $\cos^{-1}(BC \div AC)$, it becomes the projection of the auxiliary circle with its corresponding ordinates and tangents.

(*Hyperbola.* Fig. 1188.)—By (1187, 1189) the tangents at P, D meet the asymptotes in the same point L. Therefore they are parallel to CD, CP (VI. 2.)

If QT be the tangent at Q, and QV the ordinate parallel to the tangent at any other point P,

1203 $$CV \cdot CT = CP^2.$$

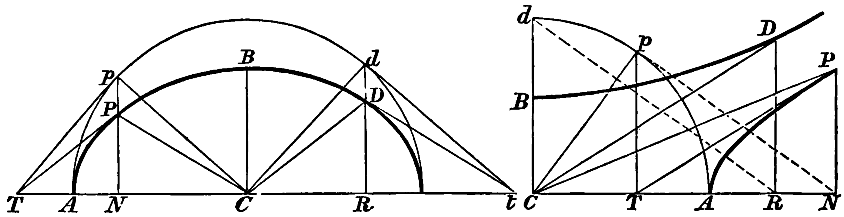


PROOF.—CR bisects PQ (1199). Therefore PW is parallel to QR. Therefore, by (VI. 2), $CV : CP = CW : CR = CP : CT$.

1204 COR.—Hence, to draw two tangents from a point T, we may find CV from the above equation, and draw QVQ' parallel to the tangent at P to determine the points of contact Q, Q'.

Let PN, DN be the ordinates at the extremities of conjugate diameters, and PT the tangent at P. Let the ordinates at N and R in the ellipse, but at T and C in the hyperbola, meet the auxiliary circle in p and d; then

1205 $$CN = dR, \quad CR = pN.$$



PROOF.—(*Ellipse.*) Cp, Cd are parallel to the tangents at d and p (Note to 1202). Therefore pCd is a right angle. Therefore pNC, CRd are equal right-angled triangles with $CN = dR$ and $CR = pN$.

(*Hyperbola.*) $CN \cdot CT = AC^2$ (1170),

and $DR \cdot CT = 2\Delta CDT = 2CDP = AC \cdot BC$ (1194);

$\therefore \quad \dfrac{CN}{DR} = \dfrac{AC}{BC} = \dfrac{pN}{PN}$ (1174), $\quad \therefore \dfrac{CN}{pN} = \dfrac{DR}{PN} = \dfrac{CR}{TN}$ (similar triangles).

But $\dfrac{CN}{pN} = \dfrac{pN}{TN}$ (VI. 8); $\therefore CR = pN$. Also $Cp = Cd$; therefore the triangles CpN, dCR are equal and similar; therefore $CN = dR$ and dR is parallel to pN.

1206 COR.— $$DR : dR = BC : AC.$$

PROOF.—(*Ellipse.*) By (1174). (*Hyperbola.*) By the similar right-angled triangles, we have $dR : pN = CR : TN = DR : PN$;

therefore $dR : DR = pN : PN = AC : BC$ (1174).

In the same figures,

1207 (*Ellipse.*) $$CN^2 + CR^2 = AC^2; \quad DR^2 + PN^2 = BC^2.$$

1209 (*Hyperbola.*) $$CN^2 - CR^2 = AC^2; \quad DR^2 - PN^2 = BC^2.$$

PROOF.—Firstly, from the right-angled triangle CNp in which $pN = CR$ (1205).

Secondly, In the ellipse, by (1174), $DR^2 + PN^2 : dR^2 + pN^2 = BC^2 : AC^2$, and $dR^2 + pN^2 = AC^2$, by (1205). For the hyperbola, take *difference* of squares.

1211 (*Ellipse.*) $CP^2+CD^2 = AC^2+BC^2$.

1212 (*Hyperbola.*) $CP^2-CD^2 = AC^2-BC^2$.

PROOF.—(Figs. 1205.) By (1205—1210) and (I. 47), applied to the triangles CNP, CRD.

The product of the focal distances is equal to the square of the semi-conjugate diameter, or

1213 $$PS.PS' = CD^2.$$

PROOF.—(*Ellipse.* Fig. 1171.) $2PS.PS' = (PS+PS')^2 - PS^2 - PS'^2 = 4AC^2 - 2CS^2 - 2CP^2$ (922, i.) $= 2(AC^2+BC^2-CP^2)$ (1164) $= 2CD^2$(1211).

(*Hyperbola.*)—Similarly with $2PS.PS' = PS^2+PS'^2-(PS'-PS)^2 =$ &c.

1214 The products of the segments of intersecting chords QOq, $Q'Oq'$ are in the ratio of the squares of the diameters parallel to them, or

$$OQ.Oq : OQ'.Oq' = CD^2 : CD'^2.$$

PROOF.—(*Ellipse.*) By projection from the circle (1088); for the proportion is true for the circle, by (III. 35, 36).

(*Hyperbola.* Fig. 1188.) Let O be any point on Qq. Draw IOi parallel to Ee, meeting the asymptotes in I and i; then

$$OR.Or - OQ.Oq = QR.Qr \text{ (II. 5)} = PL^2 \text{ (1191)} \ldots\ldots\ldots (1).$$

Now $\dfrac{OR}{OI} = \dfrac{PL}{PE}$, and $\dfrac{Or}{Oi} = \dfrac{Pl}{Pe}$; $\therefore \dfrac{OR.Or}{OI.Oi} = \dfrac{PL^2}{PE.Pe} = \dfrac{CD^2}{BC^2}$ (1184).

Therefore $\dfrac{OR.Or-PL^2}{OI.Oi-BC^2} = \dfrac{CD^2}{BC^2}$; or, by (1), $\dfrac{OQ.Oq}{OI.Oi-BC^2} = \dfrac{CD^2}{BC^2}$.

Similarly for any other chord $Q'Oq'$ drawn through O.

Therefore $OQ.Oq : OQ'.Oq' = CD^2 : CD'^2$.

1215 COR.—The tangents from any point to the curve are in the ratio of the diameters parallel to them.

For, when O is without the curve and the chords become tangents, each product of segments becomes the square of a tangent.

1216 If from any point Q on a tangent PT drawn to *any* conic (Fig. 1220), two perpendiculars QR, QL be drawn to the focal distance PS and the directrix XM respectively; then

$$SR : QL = e.$$

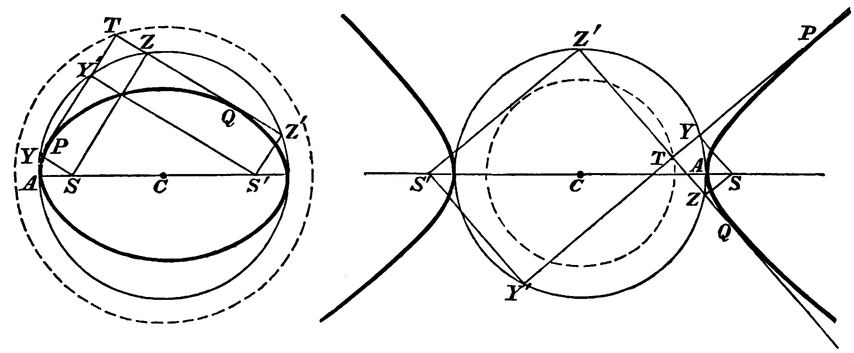
PROOF.—Since QR is parallel to ZS (1167), therefore, by (VI. 2),

$$SR : PS = QZ : PZ = QL : PM;$$

therefore $$SR : QL = PS : PM = e.$$

COR.—By applying the theorem to each of the tangents from Q, a proof of (1181) is obtained.

1217 *The Director Circle.*—The locus of the point of intersection, T, of two tangents always at right angles is a circle called the *Director Circle.*



PROOF.—Perpendiculars from S, S' to the tangents meet them in points Y, Z, Y', Z', which lie on the auxiliary circle. Therefore, by (II. 5, 6) and

(III. 35, 36), $\quad TC^2 \sim AC^2 = TZ \,.\, TZ' = SY \,.\, S'Y' = BC^2.$ (1178)

Therefore $\quad TC^2 = AC^2 \pm BC^2,$ a constant value.

NOTE.—Theorems (1170), (1177), and (1203) may also be deduced at once for the ellipse by orthogonal projection from the circle; and, in all such cases, the analogous property of the hyperbola may be obtained by a similar projection from the rectangular hyperbola if the property has already been demonstrated for the latter curve.

1218 If the points A, S (Fig. 1162) be fixed, while C is moved to an infinite distance, the conic becomes a parabola. Hence, any relation which has been established for parts of the curve which remain finite, when AC thus becomes infinite, *will be a property of the parabola.*

1219 Theorems relating to the ellipse may generally be adapted to the parabola by eliminating the quantities which become infinite, employing the principle that *finite differences may be neglected in considering the ratios of infinite quantities.*

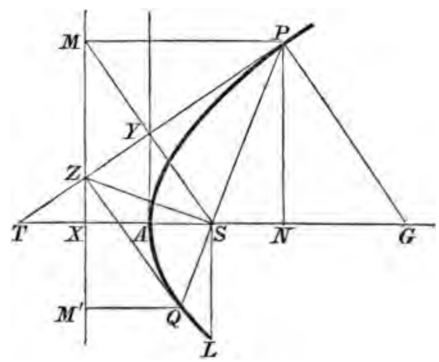
EXAMPLE.—In (1193), when P' is at infinity, VP' becomes $= 2CP$; and in (1213) PS' becomes $= 2CP$. Thus the equations become

$$\frac{QV^2}{PV} = \frac{2CD^2}{CP} \quad \text{and} \quad PS = \frac{CD^2}{2CP}.$$

Therefore $QV^2 = 4PS \,.\, PV$ in the parabola.

THE PARABOLA.

If S be the focus, XM the directrix, and P any point on the curve, the *defining property* is



1220 $PS = PM$

and $e = 1.$ (1153)

1221 Hence

$AX = AS.$

1222 The Latus Rectum $= 4AS$.

PROOF.— $SL = SX$ (1220) $= 2AS$.

1223 If PZ be a tangent at P, meeting the directrix in Z, then PSZ is a right angle.

PROOF.—As in (1167); theorem (1166) applying equally to the parabola.

1224 The tangent at P bisects the angles SPM, SZM.

PROOF.—PZ is common to the triangles PSZ, PMZ; $PS = PM$ and $\angle PSZ = PMZ$ (1223).

1225 COR.— $ST = SP = SG.$ (I. 29, 6)

1226 The tangents at the extremities of a focal chord PQ intersect at right angles in the directrix.

PROOF.—(i.) They intersect in the directrix, as in (1169).

(ii.) They bisect the angles SZM, SZM' (1224), and therefore include a right angle.

1227 The curve bisects the sub-tangent. $AN = AT$.

PROOF.— $ST = SP$ (1225) $= PM = XN$, and $AX = AS$.

1228 The sub-normal is half the latus rectum. $NG = 2AS$.

PROOF.—$ST = SP = SG$ and $TX = SN$ (1227). Subtract.

1229 $PN^2 = 4AS.AN.$

PROOF.—$PN^2 = TN.NG$ (VI. 8) $= AN.2NG$ (1227) $= 4AS.AN$ (1228). Otherwise, by (1176) and (1165); making AC infinite. See (1219).

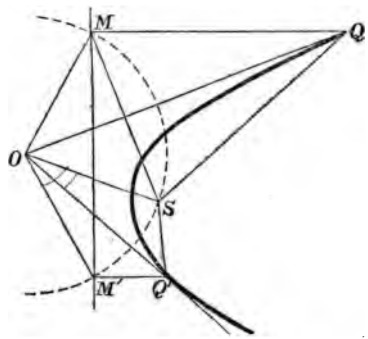
1230 The tangents at A and P each bisect SM, the latter bisecting it at right angles.

PROOF.—(i.) The tangent at A, by (VI. 2), since $AX = AS$.

(ii.) PT bisects SM at right angles, by (I. 4), since $PS = PM$ and $\angle SPY = MPY$.

1231 COR.— $SA : SY : SP.$ [By similar triangles.

1232 To draw tangents from a point O to the parabola.



CONSTRUCTION.—Describe a circle, centre O and radius OS, cutting the directrix in M, M'. Draw MQ, $M'Q'$ parallel to the axis, meeting the parabola in Q, Q'. Then OQ, OQ' will be tangents.

PROOF.— OS, $SQ = OM$, MQ (1220); therefore, by (I. 8), $\angle OQS = OQM$; therefore OQ is a tangent (1224). Similarly OQ' is a tangent.

Otherwise, by (1181). When S' moves to infinity, the circle MM' becomes the directrix.

1233 COR. 1.—The triangles SQO, SOQ' are similar, and

$$SQ : SO : SQ'.$$

PROOF.— $\angle SQO = MQO = SMM' = SOQ'.$ (III. 20)

Similarly $SQ'O = SOQ.$

1234 COR. 2.—The tangents at two points subtend equal angles at the focus; and they contain an angle equal to half the exterior angle between the focal distances of the points.

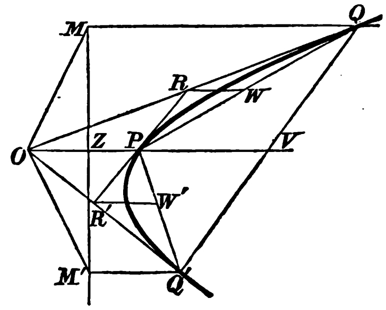
PROOF.— $\angle OSQ = OSQ'$, by (Cor. 1).

Also $\angle QOQ' = SOQ + SQO = \pi - OSQ = \frac{1}{2}QSQ'.$

1235 DEF.—Any line parallel to the axis of a parabola is called a *diameter*.

1236 The chord of contact QQ' of tangents from any point O is bisected by the diameter through O.

PROOF.—This proposition and the corollaries are included in (1198–1200), by the principle in (1218). An independent proof is as follows.



The construction being as in (1232), we have $ZM = ZM'$; therefore $QV = VQ'$ (VI. 2).

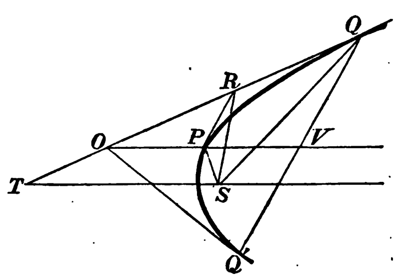
1237 Cor. 1.—The tangent RR' at P is parallel to QQ'; and $OP = PV$.

Proof.—Draw the diameter RW. $QW = WP$; therefore $QR = RO$ (VI. 2). Similarly $Q'R' = R'O$.

1238 Cor. 2.—Hence, the diameter through P bisects *all* chords parallel to the tangent at P.

If QV be a semi-chord parallel to the tangent at P,

1239 $$QV^2 = 4PS.PV.$$



This is the fundamental equation of the parabola, equation (1229) being the most important form of it.

Proof.—Let QO meet the axis in T. By similar triangles (1231),

$\angle SRP = SQR = STQ = POR$;

and $\angle SPR = OPR$ (1224). Therefore $PR^2 = PS.PO = PS.PV$ and $QV = 2PR$.

Otherwise: See (1219), where the equation is deduced from (1193) of the ellipse.

1240 Cor. 1.—If v be any other point, either within or without the curve, on the chord QQ', and pv the corresponding diameter, $$vQ.vQ' = 4pS.pv. \qquad \text{(II. 5)}$$

1241 Cor. 2.—The focal chord parallel to the diameter through P, and called the *parameter* of that diameter, is equal to $4SP$. For PV in this case is equal to PS.

1242 The products of the segments of intersecting chords, QOq, $Q'Oq'$, are in the ratio of the parameters of the diameters which bisect the chords; or

$$OQ.Oq : OQ'.Oq' = PS : P'S.$$

Proof.—By (1240), the ratio is equal to $4PS.pO : 4P'S.pO$.

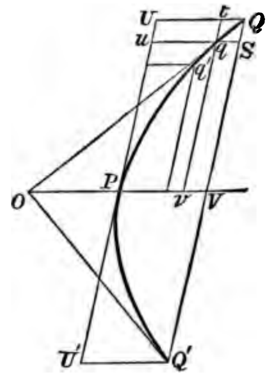
Otherwise: In the ellipse (1214), the ratio is $= \frac{CD^2}{CD'^2} = \frac{PS.PS'}{P'S.P'S'}$ (1213) $= \frac{PS}{P'S}$, when S' is at infinity and the curve becomes a parabola (1219).

1243 Cor.—The squares of the tangents to a parabola from any point are as the focal distances of the points of contact.

Proof.—As in (1215). Otherwise, by (1233) and (VI. 19).

1244 The area of the parabola cut off by any chord QQ' is two-thirds of the circumscribed parallelogram, or of the triangle formed by the chord and the tangents at Q, Q'.

Proof.—Through Q, q, q', &c., adjacent points on the curve, draw right lines parallel to the diameter and tangent at P. Let the secant Qq cut the diameter in O. Then, when q coincides with Q, so that Qq becomes a tangent, we have $OP = PV$ (1237). Therefore the parallelogram $Vq = 2Uq$, by (I. 43), applied to the parallelogram of which OQ is the diagonal. Similarly $vq' = 2uq'$, &c. Therefore the sum of all the evanescent parallelograms on one side of PQ is equal to twice the corresponding sum on the other side; and these sums are respectively equal to the areas PQV, PQU.—(Newton, Sect. I., Lem. II.)



Practical methods of constructing the Conic.

1245 *To draw the Ellipse.*

Fix two pins at S, S' (Fig. 1162). Place over them a loop of thread having a perimeter $SPS' = SS' + AA'$. A pencil point moved so as to keep the thread stretched will describe the ellipse, by (1163).

1246 *Otherwise.*—(Fig. 1173.) Draw PHK parallel to QC, cutting the axes in H, K. $PK = AC$ and $PH = BC$ (1174). Hence, if a ruler PHK moves so that the points H, K slide along the axes, P will describe the ellipse.

1247 *To draw the Hyperbola.*

Make the pin S' (Fig. 1162) serve as a pivot for one end of a bar of any convenient length. To the free end of the bar attach one end of a thread whose length is less than that of the bar by AA'; and fasten the other end of the thread to the pin S. A pencil point moved so as to keep the thread stretched, and touching the bar, will describe the hyperbola, by (1163).

1248 *Otherwise:*—Lay off any scale of equal parts along both asymptotes (Fig. 1188), starting and numbering the divisions from C, in both positive and negative directions.

Join every pair of points L, l, the *product* of whose distances from C is the same, and a series of tangents will be formed (1192) which will define the hyperbola. See also (1289).

1249 *To draw the Parabola.*

Proceed as in (1247), with this difference: let the end of the bar, before attached to S', terminate in a "T-square," and be made to slide along the directrix (Fig. 1220), taking the string and bar of the same length.

1250 *Otherwise:*—Make the same construction as in (1248), and join every pair of points, the algebraic sum of whose distances from the zero point of division is the same.

Proof.—If the two equal tangents from any point T on the axis (Fig. 1239) be cut by a third tangent in the points R, r; then RQ may be proved equal to rT, by (1233), proving the triangles SRQ, SrT equal in all respects.

1251 Cor.—The triangle SRr is always similar to the isosceles triangle SQT.

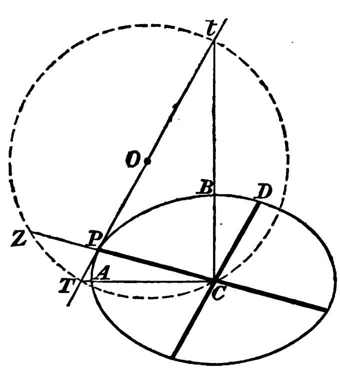
1252 *To find the axes and centre of a given central conic.*

(i.) Draw a right line through the centres of two parallel chords. This line is a diameter, by (1198); and two diameters so found will intersect in the centre of the conic.

(ii.) Describe a circle having for its diameter any diameter PP' of the conic, and let the circle cut the curve in Q. Then PQ, $P'Q$ are parallel to the axes, by (1201) and (III. 31).

1253 *Given two conjugate diameters,* CP, CD, *in position and magnitude: to construct the conic.*

On CP take $PZ = CD^2 \div CP$; measuring *from* C in the ellipse, and *towards* C in the hyperbola (Fig. 1188). A circle described through the points C, Z, and having its centre O on the tangent at P, will cut the tangent in the points where it is intersected by the axes.



Proof.—*Analysis:* Let AC, BC cut the tangent at P in T, t. The circle whose diameter is Tt will pass through C (III. 31), and will make

$CP \cdot PZ = PT \cdot Pt$ (III. 35, 36) $= CD^2$ (1197).

Hence the construction.

Circle and Radius of Curvature.

1254 Definitions.—The circle which has the same tangent with a curve at P (Fig. 1259), and which passes through another point Q on the curve, becomes the *circle of curvature* when Q ultimately coincides with P; and its radius becomes the *radius of curvature.*

1255 Otherwise.—The *circle of curvature* is the circle which passes through *three* coincident points on the curve at P.

1256 Any chord PH of the circle of curvature is called a *chord of curvature* at P.

1257 Through Q draw RQ' parallel to PH, meeting the tangent at P in R, and the circle in Q', and draw QV parallel to PR. RQ is called a *subtense* of the arc PQ.

1258 Theorem.—*Any chord of curvature* PH *is equal to the ultimate value of the square of the arc* PQ *divided by the subtense* RQ *parallel to the chord:* and this is also equal to

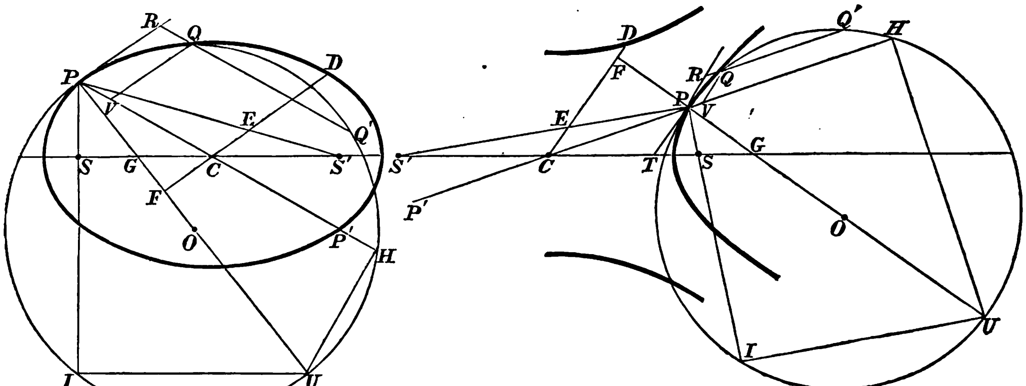
$$QV^2 \div PV.$$

Proof.—$RQ' = RP^2 \div RQ$ (III. 36). And when Q moves up to P, RQ' becomes PH; and RP, PQ, and QV become *equal* because *coincident lines.*

1259 In the ellipse or hyperbola, the semi-chords of curvature at P, measured along the diameter PC, the normal PF, and the focal distance PS, are respectively equal to

$$\frac{CD^2}{CP}, \quad \frac{CD^2}{PF}, \quad \frac{CD^2}{AC};$$

the second being the radius of curvature at P.



Proof.—(i.) By (1258), $PH = \frac{QV^2}{PV} = \frac{VP' \cdot CD^2}{CP^2}$ (1193) $= \frac{2CD^2}{CP}$ in the limit when VP' becomes $PP' = 2CP$.

(ii.) By the similar triangles PHU, PFC (III. 31), we have

$$PU \cdot PF = CP \cdot PH = 2CD^2, \text{ by (i.)}$$

(iii.) By the similar triangles PIU, PFE (1168), we have

$$PI \cdot PE = PU \cdot PF = 2CD^2, \text{ by (ii.); and } PE = AC \text{ (1179).}$$

1260 In the parabola, the chord of curvature at P (Fig. 1259) drawn parallel to the axis, and the one drawn through the focus, are each equal to $4SP$, the parameter of the diameter at P (1241).

PROOF.—By (1258). The chord parallel to the axis $= QV^2 \div PV = 4PS$ (1239); and the two chords are equal because they make equal angles with the diameter of the circle of curvature.

1261 COR.—The radius of curvature of the parabola at P (Fig. 1220) is equal to $2SP^2 \div SY$.

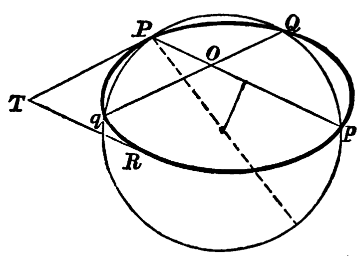
PROOF.—(Fig. 1259.) $\frac{1}{2}PU = \frac{1}{2}PI \sec IPU = 2SP \sec PSY$ (Fig. 1221).

1262 The products of the segments of intersecting chords are as the squares of the tangents parallel to them (1214–15), (1242–43).

1263 The common chords of a circle and conic (Fig. 1264) are equally inclined to the axis; and conversely, if two chords of a conic are equally inclined to the axis, their extremities are concyclic.

PROOF.—The products of the segments of the chords being equal (III. 35, 36), the tangents parallel to them are equal (1262). Therefore, by (1200).

1264 The common chord of any conic and of the circle of curvature at a point P, has the same inclination to the axis as the tangent at P.

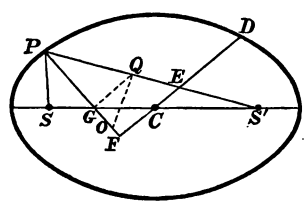


PROOF.—Draw any chord Qq parallel to the tangent at P. The circle circumscribing PQq always passes through the same point p (1263), and does so, therefore, when Qq moves up to P, and the circle becomes the circle of curvature.

1265 PROBLEM.—*To find the centre of curvature at any given point of a conic.*

First Method.—(Fig. 1264.) Draw a chord from the point making the same angle with the axis as the tangent. The perpendicular bisector of the chord will meet the normal in the centre of curvature, by (1264) and (III. 3).

1266 *Second Method.*—Draw the normal PG and a perpendicular to it from G, meeting either of the focal distances in Q. Then a perpendicular to the focal distance drawn from Q will meet the normal in O, the centre of curvature.



PROOF.—(*Ellipse or Hyperbola.*) By (1259), the radius of curvature at P

$$= \frac{CD^2}{PF} = \frac{CD^2}{BC^2} PG \text{ (1195)} = \frac{AC^2}{PF^2} PG \text{ (1194)}$$

$$= PG \sec^2 S'PG = PO.$$

For $AC = PE$ (by 1179).

(*Parabola.*) By (1261). The radius of curvature

$$= 2SP^2 \div SY = 2SP \sec SPG = PO.$$

For $2SP = PQ$, because $SP = SG$ (1125).

Miscellaneous Theorems.

1267 In the Parabola (Fig. 1239) let QD be drawn perpendicular to PV, then $QD^2 = 4AS.PV.$ (1231, 1239)

1268 Let RPR' be *any* third tangent meeting the tangents OQ, OQ' in R, R'; the triangles SQO, SRR', SOQ' are similar and similarly divided by SR, SP, SR' (1233–4).

1269 COR.— $OR.OR' = RQ.R'Q'.$

1270 Also, the triangle $PQQ' = 2ORR'.$ (1244)

With the same construction and for *any* conic,

1271 $OQ : OQ' = RQ.RP' : R'Q'.PR.$ (1215, 1243)

1272 Also the angle $RSR' = \frac{1}{2}QSQ'.$ (1181)

1273 Hence, in the Parabola, the points O, R, S, R' are concyclic, by (1234).

1274 In any conic (Figs. 1171), $SP : ST = AN : AT.$

PROOF.— $\frac{SP}{ST} = \frac{S'P}{S'T} = \frac{CA}{CT}$ (69, 1163) $= \frac{CN}{CA}$ (1170) $= \frac{AN}{AT}$ (69).

1275 COR.—If the tangent PT meets the tangent at A in R, then SR bisects the angle PST (VI. 3).

1276 In Figs. (1178), SY', $S'Y$ both bisect the normal PG.

1277 The perpendicular from S to PG meets it in CY.

1278 If CD be the radius conjugate to CP, the perpendicular from D upon CY is equal to BC.

1279 SY and CP intersect in the directrix.

1280 If every ordinate PN of the conic (Figs. 1205) be turned round N, in the plane of the figure, through the same angle PNP', the locus of P' is also a conic, by (1193). The auxiliary circle then becomes an ellipse, of which AC and BC produced are the equi-conjugate diameters.

If the entire figures be thus deformed, the points on the axis AA' remain fixed while PN, DR describe the same angle. Hence CD remains parallel to PT. CP, CD are therefore still conjugate to each other.

Hence, the relations in (1205–6) still subsist when CA, CB are any conjugate radii. Thus universally,

1281 $PN : CR = DR : CN$ or $PN . CN = DR . CR.$

1282 If the tangent at P meets any pair of conjugate diameters in T, T', then $PT . PT'$ is constant and equal to CD^2.

PROOF.—Let CA, CB (Figs. 1205) be the conjugate radii, the figures being deformed through any angle. By similar triangles,

$$\left.\begin{array}{l} PT : PN = CD : DR \\ PT' : CN = CD : CR \end{array}\right\}, \text{ therefore } PT . PT' : PN . CN = CD^2 : DR . CR.$$

Therefore $PT . PT' = CD^2$, by (1281).

1283 If the tangent at P meets any pair of parallel tangents in T, T', then $PT . PT' = CD^2$, where CD is conjugate to CP.

PROOF.—Let the parallel tangents touch in the points Q, Q'. Join PQ, PQ', CT, CT'. Then CT, CT' are conjugate diameters (1199, 1201). Therefore $PT . PT' = CD^2$ (1282).

1284 COR.—$QT . Q'T' = CD'^2$, where CD' is the radius parallel to QT.

1285 To draw two conjugate diameters of a conic to include a given angle. Proceed as in (1252 ii.), making PP' in this case the chord of the segment of a circle containing the given angle (III. 33).

1286 The focal distance of a point P on any conic is equal to the length QN intercepted on the ordinate through P between the axis and the tangent at the extremity of the latus rectum.

PROOF.—(Fig. 1220). $QN : NX = LS : SX = e$ and $SP : NX = e$.

1287 In the hyperbola (Fig. 1183). $CO : CA = e$. (1162, 1164).

If a right line PKK' be drawn parallel to the asymptote CR, cutting the one directrix XE in K and the other in K'; then

1288 $SP = PK = e . CN - AC$; $S'P = PK' = e . CN + AC.$

PROOF.—From $CR = e . CN$ (1287) and $CE = AC$ (1186).

1289 COR.—Hence the hyperbola may be drawn mechanically by the method of (1249) by merely fixing the cross-piece of the T-square at an angle with the bar equal to BCO.

1290 DEFINITION.—*Confocal* conics are conics which have the same foci.

1291 The tangents drawn to any conic from a point T on a confocal conic make equal angles with the tangent at T.

PROOF.—(Fig. 1217.) Let T be the point on the confocal conic.

$$SY : SZ = S'Z' : S'Y' \quad (1178).$$

Therefore ST and $S'T$ make equal angles with the tangents TP, TQ; and they also make equal angles with the tangent to the confocal at T (1168), therefore &c.

1292 In the construction of (1253), PZ is equal to half the chord of curvature at P drawn through the centre C (1259).